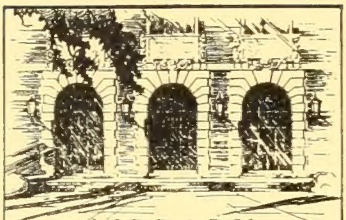

HIGH SCHOOL MATHEMATICS

Teachers' Edition

UNIT 7

University of Illinois Committee on School Mathematics

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TEACHERS COMMENTARY

Introduction

As we have indicated in the Preface [page 7-iii], Unit 7 continues the deductive organization, begun in Unit 2, of the student's intuitive knowledge of the real numbers. By the end of Unit 7, the student will have all but one of those basic principles which are needed to characterize the structure of the real number system. [The additional principle --a completeness principle--will be introduced in Unit 9.] So, as in Unit 2 and in contrast with Units 3, 4, and 5, the principal activity of Unit 7 is that of proving theorems. In particular, students will practice proving generalizations concerning the positive numbers, the inequality relations, the positive integers, and the integers. Since proofs of many generalizations about the positive integers [and the integers] require the use of mathematical induction and the notion of recursive definition, proof-by-mathematical-induction is central to much of the unit. As in Unit 6, stress is placed on column proofs and paragraph proofs and, here, the rules of reasoning and the techniques developed in the Appendix of Unit 6 are of basic importance. Those rules and techniques are further developed to embrace procedures for dealing with restricted quantifiers. [See, in particular, the COMMENTARY for pages 7-36 through 7-40.]

As in earlier units, the proofs given in the text and in the COMMENTARY include references ["Theorem 33"] to previously proved theorems which are more explicit than are to be expected ["theorem"] from students.

As you have learned from your use of earlier units, most expositions of mathematics require, for understanding, several readings separated by periods of rumination. You should call this fact to the attention of your students. If they fail to understand a sentence, the first thing to do is to read the next two sentences to see if an explanation is forthcoming. If the sentence still offers difficulty, concentrate on it and the surrounding text for a reasonable time. If, as may not infrequently happen, this does not result in complete understanding, go on to something else--but, remember to return to the subject from time to time until mastery is achieved.

As in previous units, there is considerable optional material in Unit 7. You should acquaint yourself with it, and treat it in class to the extent that time and interest will permit. In some cases, this may be assigned to one or two students and discussed out of class or presented as a project.

TEACHER'S COMMENTARY

Introduction

As we have indicated in the Preface [page [illegible]], Unit 7 continues the deductive organization, begun in Unit 2, of the students' knowledge of the real numbers. By the end of Unit 7, the student will have all but one of those basic principles which are used to characterize the structure of the real number system. [The [illegible] principle --a completeness principle [illegible].] [illegible] Unit 2 and in contrast with Units 3, 4, and 5, the main activity of Unit 7 is that of proving theorems. In particular, [illegible] proving generalizations concerning the positive numbers, the negatives, the positive integers, and the integers. [illegible] generalizations about [illegible] [illegible] purpose of mathematical induction [illegible] proof by mathematical induction [illegible]. In Unit 6, stress is placed on absolute values and inequalities, and there, the rules of [illegible] are of much importance. These rules are [illegible] developed to [illegible] manipulations. [See [illegible] the COMMENTARY for pages 7-36 through 7-40.]

As in earlier units, the [illegible] given in the text and in the COMMENTARY include references [illegible] which [illegible] notes.

As you have learned, [illegible] of [illegible] your students. [illegible] If they fail [illegible] read [illegible] text [illegible] not [illegible] remember [illegible] understanding [illegible].

[illegible] 7.7. You should [illegible] extent that [illegible] assigned [illegible] one or two [illegible] as [illegible] assignments.

Also, as in previous units, we have included more exercise material than can reasonably be assigned to students who hope to complete our program in high school. Consequently, choices have to be made not only of what should be assigned for out-of-class work but also of what should be discussed in class. The following paragraphs contain suggestions about teaching a "minimum" course in Unit 7.

Since a good deal of the theoretical development of the content is embodied in exercises, you cannot effectively save time just by omitting all lengthy exercises. What one must do to save time is to "lecture" on certain important exercises. From our point of view of discovery, this is not the most effective way to promote learning. But, if you are pressed for time, you must strike a balance between lecturing and discovery.

Another way to speed things up is to devote a lot of class time to individual seat-work. This will enable you to give attention to specific study problems. For example, you can help certain students learn how to read the text. In planning such a lesson, you need to look for points which all students should hear about. Then, at appropriate times when most students are ready for it, you can request their attention, discuss the point in question, and then have them return to their individual work. For example, the material on pages 7-3 through 7-9 can easily be handled in one day's class plus one assignment. Two points which should be discussed are what an addition or multiplication table looks like, and what to do with a column proof such as the one in Part C on page 7-6. If you discover as you move around the class that most students have forgotten the details of functional notation when they get to Part B on page 7-6, you may want to make a remark about that. It is also helpful if students have access to an answer list when they are doing manipulation exercises like those in Part A on page 7-5 and in Part B on pages 7-8 and 7-9. You can put answers on the blackboard or mimeograph them.

It is seldom necessary to assign all of the manipulation exercises to each student. Sometimes it is helpful to assign them over a period of several days. Use care in doing this, however, because we frequently arrange such exercises in sequences which promote discoveries and you would be thwarting this plan if you broke such sequences.

Our miscellaneous exercises are truly miscellaneous. They can be assigned, practically at any time, although some content of Unit 7 is incorporated in later sets. Once again, prepared answer lists will make it possible for students to check their work without help from you, and will also give you a chance to spot students who have difficulties which they cannot resolve by themselves. [Naturally, you will want to

teach students how to make correct use of answer lists. For example, in drill work, it is probably better to check at frequent intervals rather than at the very end of the assignment in order to avoid the danger of fixing wrong procedures.]

Students can read pages 7-12 and 7-13 on their own, but should see a sample "long division" problem worked out step-by-step, especially one in which terms are missing in the dividend. Exercise B3 should be discussed in class.

Pages 7-16 and 7-17 can be read independently, but the discussion following the display on page 7-17 should be reviewed in class. Be sure to discuss the notational change described on page 7-19.

Students should read aloud pages 7-22 through 7-24, with you constantly stressing the point that the basic principles (P_1) - (P_4) are not supposed to be startling revelations but simply codifications of the student's experience. They are supposed to be "obvious" things, as are most of the theorems in the exercises which follow. But, the motivation for proof here is not to gain conviction, but to show that lots of things whose obviousness came about as a result of experiments with numbers could have been predicted by deductive reasoning. The ulterior motive is to give students an opportunity to practice various proof strategies. The exercises on pages 7-24 through 7-27 are to provide drill in deductive reasoning. One would certainly not assign all of them to any student. Moreover, it would be foolhardy to strive for mastery of these pages before moving ahead. Instead, plan your assignments so that students can have further opportunities to practice with these exercises. It is difficult to organize a textbook to cater to both logical and pedagogical arrangements. Logically, the exercises on these pages belong together, but pedagogically, they should be attacked in a spiral fashion. The teacher must take responsibility for the pedagogical arrangement. This is seldom a major concern in using textbooks for the "traditional" curriculum, because such books deal primarily with manipulation and problems, and although there is an ordering for skills, it is quite crude.

Similar remarks about a spiral treatment apply to pages 7-30 through 7-42. The very minimum to be expected of all students is that they agree that each of the theorems treated therein is "obvious" and is predictable from the fifteen basic principles together with basic principle (G). They must also understand that the major contribution of (G) is to link up properties of > with those of positiveness. The geometrical approach on pages 7-38 and 7-39 is designed to contribute to the "obviousness" of the theorems on pages 7-40 and 7-41. Students also need manipulative

practice in applying these theorems in solving inequations; hence, Part I and the inequations scattered throughout the miscellaneous exercises.

The content of pages 7-45 through 7-59 is very important--but, once again, the initial exposure to what is there can be brief since it should be just the first of a sequence of exposures. As students work on inductive proofs, you will have many opportunities to bring in the terminology introduced on page 7-53, and more important, deal with the logical matters revealed by the tree-diagram on page 7-65.

The matter of recursive definition is also important. The exercises on pages 7-66 through 7-70 serve to bring home the concept of recursive definition and should be handled completely rather than in a spiral fashion. On the other hand, the exercises on pages 7-71 through 7-75 involve just an application of recursive definition and could be treated later in the course. However, they, too, must be treated as a unit and, since they are preparation for the work on combinations and permutations in Unit 8, they cannot be omitted. [See, also, Miscellaneous Exercise 1 on page 7-91.]

The material on pages 7-84 through 7-91 can be treated in the way we suggested treating section 7.03 on inequations. Pages 7-88 and 7-89 should be read aloud and discussed in class. [Part B on page 7-90 is important preparation for the topic of sums of infinite sequences which comes up in Unit 8.]

Pages 7-94 and 7-95 should be read in class, the exercises on 7-96 and 7-97 assigned for homework, and the material on 7-98 through 7-100 discussed in class the next day. [Theorems 114, 116, and 117 are referred to in Unit 8.]

The notion of the greatest integer function [pages 7-102 through 7-107] should not be new to your students. They should be able to carry out computations like those in Exercise A2 on page 7-103, and, at the very least, follow some of the proofs for Exercise A3, and the proof in the text on pages 7-104 and 7-105.

The work on pages 7-115 and 7-119 should be handled in one class discussion and one assignment.

This completes the minimum course [except for the Review Exercises].

Although Part D on pages 7-131 and 7-132 is in an optional section, all students should be directed to look at it. Perhaps some will be sufficiently stimulated [by the idea that such problems can be solved

in a systematic fashion] to study the preceding sections on their own. In fact, of all the optional material in this unit, Part D is likely to appeal most to your students. In a just-more-than-minimum course you might work up to this in the following way. First, do the Exploration Exercises beginning on page 7-125 in class. These exercises are meant to lead to the discovery of Theorem 129 on page 7-129. Students should, with your help, discover the significance of the highest common factor of the coefficients of a linear equation while doing the exercises. Next, read pages 7-127 and 7-128 and Theorem 129. The reference, near the bottom of page 7-128, to the Euclidean algorithm may whet some students' interest to the point of studying pages 7-119 through 7-124, but the bracketed remark which follows the reference points out that it is not necessary to do this. Students should understand Theorem 128 as well as Theorem 129, but, in the situations being discussed, the proofs should be omitted. Continue from the '* * *' on page 7-130.

In augmenting the minimal course described above, spend more time on sections 7.03, 7.04 [through page 7-75], 7.05, and the optional material in section 7.06. Arrange your assignments so that, for each section, the extra time is spread out. Return to each subject as often as possible rather than try to achieve mastery of each before proceeding to the next.

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There is an excellent film on mathematical induction produced by the Mathematical Association of America. In two 30-minute reels, Professor Leon Henkin presents this topic with clarity and humor. The reels may be rented from the Audio-Visual Center, University of Buffalo, Buffalo 14, New York. Refer to them by 'Henkin: Reels 1 and 2, Mathematical Induction'.

VII. 126 This can be solved by counting routes. However, it is a typical combinatorial problem of the kind which can be solved by finding, for some positive integers m and n, the number of m-membered subsets of an (m + n)-membered set. [In this case, m = 5 and n = 4.] In general, if Z.H.S. were m blocks west and n blocks south of Milton's home, the number of available routes, each m + n blocks long, would be the number of m-membered subsets of an (m + n)-membered set--that is, it would be

$$\frac{(m+n)!}{m!n!}$$

[You can see the relation of Milton's problem to the subset problem in this way. Suppose that Milton lives on the corner of Central Avenue [N-S] and Division Street [E-W]. The N-S streets west of his home are 1st Avenue, 2nd Ave., etc., and the E-W streets south of his home are 1st St., 2nd St., etc. In particular, Z.H.S. is on the corner of 5th Avenue and 4th Street. In describing the route pictured in the figure, Milton says:

> I walked to 1st Ave., then to 1st St., then to 2nd Ave., then to 3rd Ave., then to 2nd St., then to 4th Ave., then to 3rd St., then to 4th St., and, finally, to 5th Ave.

He keeps track of the routes he takes, from day to day, by putting check-marks on a sheet of paper ruled into 9 columns, corresponding, in succession, to the 9 choices he makes in picking a route. His record of the trip just described looks like this:

✓		✓	✓		✓			✓

For each route he takes, there are five check-marks, which correspond, in order, to his decisions to go west to 1st Ave., 2nd Ave., 3rd Ave., 4th Ave., and 5th Ave., respectively, and four blanks, which correspond, in order, to 1st St., 2nd St., 3rd St., and 4th St., respectively. So, each route corresponds to a 5-membered subset of a 9-membered set. Moreover, for each arrangement of five check-marks there is a route, and for different routes he has different arrangements of check-marks. Consequently, the number of routes is the same as the number of five-membered subsets of a 9-membered set.]

III. 44; 500 Students will probably discover that, for each positive integer n, the sum of the first n odd positive integers is n^2. This is a theorem from the theory of arithmetic progressions. They will prove this theorem in Unit 8. [It is the basis of a standard method for using a desk calculator to compute approximations to square roots.]

IV. 1998; 1999 Students may guess that, for each positive integer n, the sum of the reciprocals of the principal square roots of the first n positive integers is between $2\sqrt{n} - 2$ and $2\sqrt{n} - 1$. This is an example of a kind of theorem which they will discover and prove in Unit 8. In discussing the last part of the problem you might bring out that the sum in question is

$$1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} + \frac{1}{2} + \frac{\sqrt{5}}{5}.$$

From the given data [and the irrationality of $\sqrt{2}$] it follows that

$$1.4135 < \sqrt{2} < 1.4145.$$

So,

$$\frac{1.4134}{2} < \frac{\sqrt{2}}{2} < \frac{1.4146}{2},$$

and

$$0.7067 < \frac{\sqrt{2}}{2} < 0.7073.$$

Using similarly obtained estimates for $\sqrt{3}/3$ and $\sqrt{5}/5$ one finds that the sum in question lies between 3.2309 and 3.2321. Since $2\sqrt{5} - 2 < 2.473$ and $2\sqrt{5} - 1 > 3.471$, it follows that the sum in question is between $2\sqrt{5} - 2$ and $2\sqrt{5} - 1$.

V. In Unit 8, students will prove:

$$\forall_{x > 0} \forall_{n > 1} (1 + x)^n > 1 + nx$$

[This is a sharpened form of Bernoulli's Inequality.] So, in particular, $(1.01)^{100} > 1 + 1$ and $(1.05)^{100} > 1 + 5$.

VI. This problem foreshadows the binomial theorem, two consequences of which are:

$$\forall_x \forall_y (x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

and:

$$\forall_x \forall_y (x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

The seven problems in the introduction are similar in that each is easy to solve by using an appropriate generalization about positive integers whose proof requires the use of mathematical induction. Mathematical induction will be studied in the present unit [see section 7.04], and the generalizations relevant to the first six problems will be established in Unit 8. [Problem VII is discussed on page 7-74.] Each of the problems can be solved by straight-forward computation, but the extent of the computation required for problems III, IV, and V makes this method of solution impractical. However, as suggested in the COMMENTARY on each of these problems, students can probably guess the correct solutions.

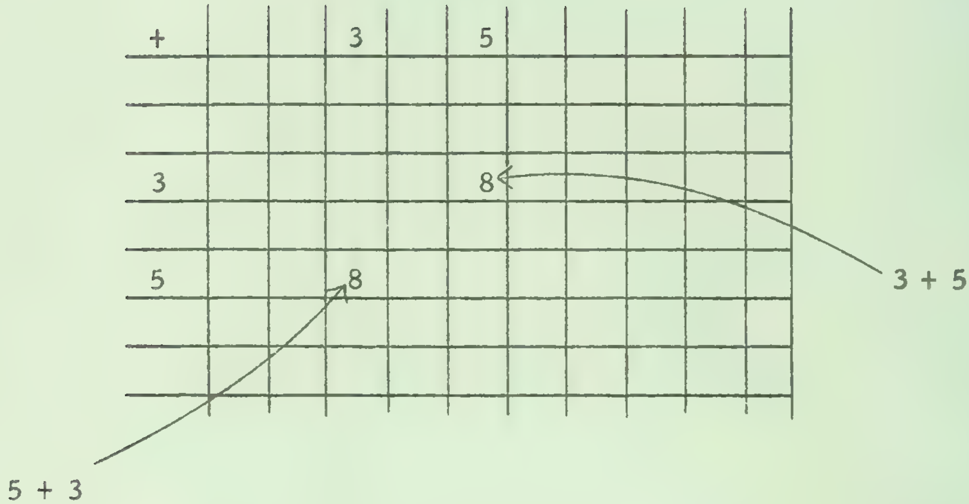
Since the purpose of these problems is merely to whet student's curiosity, do not spend a great amount of time discussing them. One procedure is to make these part of the first assignment, telling students that there will be a short class-discussion of each problem at such times as there are students ready with solutions or, in the case of the more difficult problems, relevant comments.

*

Answers to problems.

I. The first position pays more money. It is easy to solve this problem by computing the amounts paid. The result--that the first position will pay \$24500 and the second only \$24000--is usually a surprise. However, after a little thought, one sees that for the first position [as, obviously, for the second] the yearly rate of pay increases \$400 each year. So, the differences in total payments is accounted for by the fact that, since the first position pays \$100 more for the first year, it pays \$100 more during each of the five years. Your students may find it interesting to discover what annual increase will result in the second position paying the same total amount in five years as does the first. [Answer: \$450] The investigation of generalizations of this problem is aided by a knowledge of the theory of arithmetic progressions--one of the topics studied in Unit 8.

II. The second job pays \$10734418.23 more than the first. This problem, like problem I, can be solved by computation. Of course, the theory of geometric progressions, taken up in Unit 8, would cut down the work required.


+
3
5
3
8
5
8
3 + 5
5 + 3

The discussion on page 7-3 suggests how to go about justifying the entries in a, say, 9-by-9 addition table. The entries in the 1-column,

+	1	2	3	4	5	6	7	8	9
1	2								
2	3								
3	4								
4	5								
5	6								
6	7	8	9	10	11	12	13	14	15
7	8								
8	9								
9	10								

except for that in the 9-row, merely record definitions. And, as previously pointed out, $9 + 1 = 1 \cdot (9 + 1) + 0 = 10$ by virtue of the cpm, pm1, pa0, and the definition of place value.

As examples, the additional entries in the 6-row can be justified as follows:

$$6 + 2 = 6 + (1 + 1) = (6 + 1) + 1 = 7 + 1 = 8$$

$$6 + 3 = 6 + (2 + 1) = (6 + 2) + 1 = 8 + 1 = 9$$

$$6 + 4 = 6 + (3 + 1) = (6 + 3) + 1 = 9 + 1 = 10$$

$$6 + 5 = 6 + (4 + 1) = (6 + 4) + 1 = 10 + 1 = 1 \cdot 10 + 1 = 11$$

$$\begin{aligned} 6 + 6 &= 6 + (5 + 1) = (6 + 5) + 1 = (1 \cdot 10 + 1) + 1 \\ &= 1 \cdot 10 + (1 + 1) = 1 \cdot 10 + 2 = 12 \end{aligned}$$

$$\begin{aligned} 6 + 7 &= 6 + (6 + 1) = (6 + 6) + 1 = (1 \cdot 10 + 2) + 1 \\ &= 1 \cdot 10 + (2 + 1) = 1 \cdot 10 + 3 = 13 \end{aligned}$$

$$\begin{aligned} 6 + 8 &= 6 + (7 + 1) = (6 + 7) + 1 = (1 \cdot 10 + 3) + 1 \\ &= 1 \cdot 10 + (3 + 1) = 1 \cdot 10 + 4 = 14 \end{aligned}$$

$$\begin{aligned} 6 + 9 &= 6 + (8 + 1) = (6 + 8) + 1 = (1 \cdot 10 + 4) + 1 \\ &= 1 \cdot 10 + (4 + 1) = 1 \cdot 10 + 5 = 15 \end{aligned}$$

Students should see an addition table on the blackboard and become accustomed to the order convention as illustrated on the next page.

as those above. For example, the principle for adding 0 implies that $(1 + 1) + 0$ is $(1 + 1)$, and the associative principle for addition has as one consequence the theorem:

$$(*) \qquad (1 + 1) + (1 + 1) = [(1 + 1) + 1] + 1$$

It is customary [and very convenient] to supplement the basic principles by definitions of "standard numerals" for certain real numbers. Thus, one introduced '2' as an abbreviation for '1 + 1', '3' for '(1 + 1) + 1', etc. In the usual decimal notation the 'etc.' carries on to:

'9' for '(((((((1 + 1) + 1) + 1) + 1) + 1) + 1) + 1) + 1',

and then yields to a definition of place value. By the definition of place value, '10' is an abbreviation for '$1 \cdot (9 + 1) + 0$'. By the cpm, the pm1, and the pa0, this new definition yields the theorem '10 = 9 + 1'. [The question of the existence of a place value decimal numeral for each positive integer is explored in optional material on pages 7-107 through 7-111. The proof that, for each positive integer $b > 1$, each positive integer has a base-b representation, is given in optional exercises in Unit 8.]

Using the definitions of '2', '3', and '4', the theorem (*) can be abbreviated to:

$$2 + 2 = 4$$

This last statement is, as is shown on page 7-3, a consequence of definitions and an important special case:

$$\forall_x \forall_y \; x + (y + 1) = (x + y) + 1$$

of the associative principle for addition. [The role of this special case of the apa is brought out more clearly in optional exercises on pages 7-77 through 7-79.]

A proof that 2 + 9 is 11 is given at the bottom of page 7-3. It makes use of the theorem '2 + 8 = 10', presumed to have already been proved, and the theorem '$1 \cdot 10 + 1 = 11$', a consequence of the definition of place value.

*

In earlier units students selected [with the help of the textbook] ten basic principles [see page 7-17] as a basis for organizing deductively some of their knowledge of the real numbers. They also used, without proof, computing facts such as that $3 + 5 = 8$, that $6 \cdot 3 = 18$, that $2 \neq 0$, etc. In section 7.01 students will see how some of these computing facts fit into the deductive organization based on the ten principles. And, they will see the need for adopting further basic principles in order to take account of the remaining computing facts. [There is one new basic principle, '$1 \neq 0$', in section 7.01 and there are four more, (P_1) - (P_4), in section 7.02.]

[See Part II of Denbow's article "To Teach Modern Algebra" in the March 1959 issue of The Mathematics Teacher for an enlightening discussion of the difference between what Denbow calls 'a pragmatic postulational system' and 'an abstract postulational system'. The UICSM point of view toward secondary school mathematics is that it should be pragmatically organized.]

Besides augmenting the theoretical basis for their knowledge of the real numbers, students will, in section 7.01, practice the standard addition, multiplication, and division-with-remainder algorithms of elementary algebra.

*

The theorems proved in Unit 2 and those to be proved in the present unit are given on pages 7-145 through 7-154. These pages also contain the basic principles from which the theorems are derived. Since the new principle '$1 \neq 0$' is stated on page 7-145, it would be well not to attract attention to this page until you have completed the discussion of page 7-19. At that time it may be convenient to read over the first 78 theorems [those from Unit 2]. This will have the advantage of giving practice in reading the '$\Longrightarrow$' and '$\Longleftrightarrow$' notation which is introduced on page 7-19.

*

The only real numbers which are mentioned explicitly in the basic principles are 0, "the identity element for addition", and 1, "the identity element for multiplication". However, the language of the basic principles does provide for numerals other than '0' and '1'--for example:

$$-1,\ (1 + 1) + 0,\ (1 + 1) + 1,\ (1 + 1) + (1 + 1),\ [(1 + 1) + 1] \div (1 \cdot 1)$$

The basic principles [and the theorems derived from them] make assertions about all real numbers, including those named by numerals such

One of the advantages of place value notation is that it provides a foundation for simple addition and multiplication algorithms. [You might expand on this by, as a joke, asking students to simplify 'XLVI + XXIV' and 'XXI • IV'.]

As is pointed out on page 7-4, one <u>could</u> arrive at a proof of, say, '54 + 368 = 422' by justifying, in succession, each of the first 368 entries in the 54-row of an addition table. Alternatively, since, by the cpa, 54 + 368 is 368 + 54, one could justify the first 54 entries in the 368-row. Fortunately, the addition algorithm gives a simpler procedure.

In the case of '675 + 237 = 912' the algorithm is justified, in the manner of Unit 2, as follows:

$$\begin{aligned}
& 675 + 237 \\
&= (6 \cdot 10^2 + 7 \cdot 10 + 5) + (2 \cdot 10^2 + 3 \cdot 10 + 7) && \text{definition} \\
&= (6 + 2) \cdot 10^2 + (7 + 3) \cdot 10 + (5 + 7) && \text{apa, dpma, cpa} \\
&= (6 + 2) \cdot 10^2 + (7 + 3) \cdot 10 + 12 && \text{addition table} \\
&= (6 + 2) \cdot 10^2 + (7 + 3) \cdot 10 + (1 \cdot 10 + 2) && \text{definition} \\
&= (6 + 2) \cdot 10^2 + [(1 + 7) + 3] \cdot 10 + 2 && \text{apa, dpma, cpa} \\
&= (6 + 2) \cdot 10^2 + 11 \cdot 10 + 2 && \text{addition table} \\
&= (6 + 2) \cdot 10^2 + (1 \cdot 10 + 1) \cdot 10 + 2 && \text{definition} \\
&= (6 + 2) \cdot 10^2 + (1 \cdot 10^2 + 1 \cdot 10) + 2 && \text{dpma, apm, } 10 \cdot 10 = 10^2 \\
&= [(1 + 6) + 2] \cdot 10^2 + 1 \cdot 10 + 2 && \text{apa, dpma, cpa} \\
&= 9 \cdot 10^2 + 1 \cdot 10 + 2 && \text{addition table} \\
&= 912 && \text{definition}
\end{aligned}$$

"Put down the 2 and carry the 1."

More graphically:

$675 =$	$6 \cdot 10^2 +$	$7 \cdot 10 +$	5	[definition]
$237 =$	$2 \cdot 10^2 +$	$3 \cdot 10 +$	7	[definition]
			$5 + 7$	[cpa, apa]
			12	[addition table]
			$1 \cdot 10 + 2$	[definition]
		$[(1 + 7) + 3] \cdot 10 +$	2	[apa, dpma, cpa]
		$11 \cdot 10 +$	2	[addition table]
		$(1 \cdot 10 + 1) \cdot 10 +$	2	[definition]
		$1 \cdot 10^2 + 1 \cdot 10 +$	2	[dpma, apm, $10 \cdot 10 = 10$
	$[(1 + 6) + 2] \cdot 10^2 +$	$1 \cdot 10 +$	2	[apa, dpma, cpa]
	$9 \cdot 10^2 +$	$1 \cdot 10 +$	2	[addition table]
$675 + 237 = 912$				[definition]

*

When students originally learned the addition table and the associated addition algorithm, the numerals involved were names of finite cardinals. Later, the same numerals were used as names for the corresponding whole numbers of arithmetic. The table and algorithm continued to work because the finite cardinals and the whole numbers of arithmetic are isomorphic with respect to addition. In Unit 1, students learned that the numbers of arithmetic and the nonnegative real numbers are isomorphic with respect to addition. So, the table and algorithm continue to apply when the numerals are, as here, interpreted as names for nonnegative real integers.

In Unit 1 students learned how to add real numbers by adding or subtracting the numbers of arithmetic which are their absolute values. [Rules for adding real numbers were formulated in Unit 2, pages 2-28 and 2-29.] To implement this procedure one needs an algorithm for subtracting a smaller positive integer from a larger one. This algorithm is, of course, the same as the one students learned for subtracting a smaller finite cardinal from a larger one. Although we have not thought it necessary to include a discussion of this algorithm in the text,

you may be questioned on it. As an aid in answering such questions we give a justification of the algorithmic proof of '625 - 237 = 388'.

$$
\begin{aligned}
& 625 - 237 \\
&= (6 \cdot 10^2 + 2 \cdot 10 + 5) - (2 \cdot 10^2 + 3 \cdot 10 + 7) && \} \text{ definition} \\
&= (6 \cdot 10^2 + [1 + 1] \cdot 10 + 5) - (2 \cdot 10^2 + 3 \cdot 10 + 7) && \} \text{ definition} \\
&= (6 \cdot 10^2 + 1 \cdot 10 + [1 \cdot 10 + 5]) - (2 \cdot 10^2 + 3 \cdot 10 + 7) && \} \text{ dpma, apa (twice)} \\
&= (6 \cdot 10^2 + 1 \cdot 10 - [2 \cdot 10^2 + 3 \cdot 10]) + ([1 \cdot 10 + 5] - 7) && \} \text{ Th. 46} \\
&= (6 \cdot 10^2 + 1 \cdot 10 - [2 \cdot 10^2 + 3 \cdot 10]) + (15 - 7) && \} \text{ definition} \\
&= (6 \cdot 10^2 + 1 \cdot 10 - [2 \cdot 10^2 + 3 \cdot 10]) + 8 && \} \text{ addition table} \\
&= [(6 \cdot 10 + 1) \cdot 10 - (2 \cdot 10 + 3) \cdot 10] + 8 && \} \text{ apm, dpma} \\
&= [6 \cdot 10 + 1 - (2 \cdot 10 + 3)] \cdot 10 + 8 && \} \text{ dpms} \\
&= [(5 + 1) \cdot 10 + 1 - (2 \cdot 10 + 3)] \cdot 10 + 8 && \} \text{ definition} \\
&= [5 \cdot 10 + (1 \cdot 10 + 1) - (2 \cdot 10 + 3)] \cdot 10 + 8 && \} \text{ dpma, apa} \\
&= [(5 \cdot 10 - 2 \cdot 10) + ([1 \cdot 10 + 1] - 3)] \cdot 10 + 8 && \} \text{ Th. 46} \\
&= [(5 \cdot 10 - 2 \cdot 10) + (11 - 3)] \cdot 10 + 8 && \} \text{ definition} \\
&= [(5 \cdot 10 - 2 \cdot 10) + 8] \cdot 10 + 8 && \} \text{ addition table} \\
&= (5 \cdot 10 - 2 \cdot 10) \cdot 10 + 8 \cdot 10 + 8 && \} \text{ dpma} \\
&= (5 - 2) \cdot 10^2 + 8 \cdot 10 + 8 && \} \text{ dpms, apm, } 10 \cdot 10 = 10^2 \\
&= 3 \cdot 10^2 + 8 \cdot 10 + 8 && \} \text{ addition table} \\
&= 388 && \} \text{ definition}
\end{aligned}
$$

*

The addition and subtraction algorithms can, as students know, be extended to apply to decimal representations of rational real numbers [and the same can be said of the multiplication and division-with-remainder algorithms]. The algorithms for addition, subtraction, multiplication, and division of numbers which are represented by fractions are justified by Theorems 57, 58, 59, and 73, respectively.

It is relevant to point out here that all these algorithms are merely short cuts for "simplifying" numerical expressions. For example, when one asks for the sum of 675 and 237, he means to ask for more than this [if not, he should be satisfied with the answer '675 + 237']. What he wants is the decimal name for the sum. The addition algorithm provides a short cut for simplifying '675 + 237' to the desired answer, '912'. So, instead of saying 'the addition algorithm' it would be more descriptive to say 'the algorithm for transforming an indicated sum with decimal numerals into a decimal numeral'. Similarly, when one is asked 'What is the sum of 1/2 and 1/3?' a literally correct answer is '1/2 + 1/3'. But his questioner probably expects him to simplify this answer to '5/6'.

*

The main reason for bringing up the addition algorithm is to furnish the familiar analogue for the algorithm exemplified in Examples 1 and 2 on pages 7-4 and 7-5. This latter algorithm is, of course, somewhat simpler than the former in that it involves no carrying maneuver. It is somewhat more complicated in that it involves subtraction as well as addition. In Example 1, one begins by replacing, mentally '- 7' and '- 9' by '+ −7' and '+ −9', respectively. Since $-7 + -9 = -(7+9) = -16$, one could write, at the end of the third line of the solution, '+ −16'. Instead, one writes '- 16'. Thus, one uses the ps twice, then Theorem 18, and, finally the ps again.

The explanation asked for in connection with Example 2 is that, by the ps, one can replace, in the example, '$-(b - 3a - 3c)$' by '$+ -(b - 3a - 3c)$', and that '$-(b - 3a - 3c)$' is equivalent to '$(3a - b + 3c)$'. [Students need not give a justification of this equivalence--but, one way to do so is by appealing to Theorems 33 and 36, the cpa, and Theorem 37.]

*

The algorithm illustrated in Examples 1 and 2 furnishes a pedestrian procedure for simplifying algebraic expressions of the appropriate kind. Once the algorithm is understood, students should practice the more efficient sight procedure. For example, when asked to simplify:

$$(a + 3b - c) + (a - 2c) - (b - 3a - 3c)$$

one may say to oneself:

a plus a is 2a, plus 3a

and write:

Then: $5a + 2b$

3b plus op of b

and write:

Finally:

op of c plus op of 2c is op of 3c, plus 3c is 0

and write nothing.

After a little of this sort of practice, one learns to say nothing and merely writes the answer.

*

Answers for Part A.

1. $10a + 13b$	2. $12x - 3y$	3. $-10t - 4s$
4. $-6c$	5. $5y$	6. $-3y + 3z$
7. $8a - 10b + 5c$	8. $13x^2 + 6x - 5$	9. $2y^2 + 14y - 1$
10. $9k + 15m + n$	11. $3t^2 - 11t + 12$	12. 0
13. $4a$	14. $8x^2 - 9x - 2$	15. $3t^3 + 4t^2 + t + 16$
16. $4s^4 + 4s^2 + 27$	17. $4a^2 + 3ab$	18. $6y + 6$
19. $-6x^3 - 2x^2 - 1$	20. $4a^3 - 6a^2b + 14ab^2 + 9b^3$	

The custom of describing a function by a single sentence such as '$f(x) = 3x^2 - 5x + 7$' leads to a peculiar form of statement for exercises like those in Part B. Of course, in view of their work in Unit 5, students will have no trouble understanding what is required. But, you may find it interesting and helpful to know various correct interpretations. One such interpretation, say of the Sample, is that one is to complete the following to obtain a true statement:

$$(*) \qquad \forall_f\, [\forall_x\, f(x) = 3x^2 - 5x + 7 \Longrightarrow \forall_a\, f(a) + f(3a) = \qquad\qquad]$$

An <u>incorrect</u> interpretation of this exercise is that one is to complete the following to obtain a true statement:

$$\forall_f\, \forall_x\, \forall_a\, [f(x) = 3x^2 - 5x + 7 \Longrightarrow f(a) + f(3a) = \qquad\qquad]$$

[If one writes '$30a^2 - 20a + 14$' in the blank above, the resulting statement is equivalent to:

$$\forall_f\, [\exists_x\, f(x) = 3x^2 - 5x + 7 \Longrightarrow \forall_a\, f(a) + f(3a) = 30a^2 - 20a + 14]$$

which is false and is not the conclusion the authors intended the students to reach.] But, note well, that it is only because the sentence of the Sample occurs as an <u>exercise</u> that it is correctly interpreted by (*). If the completed sentence:

$$(**) \quad f(x) = 3x^2 - 5x + 7 \Longrightarrow f(a) + f(3a) = 30a^2 - 20a + 14$$

occurred at the end of a test-pattern, it would be proper to generalize. [See, for example, steps (11) and (12) of the proof on page 7-64.] The intent of the exercise is that, for the function defined by:

$$f(x) = 3x^2 - 5x + 7$$

one is to prove the generalization:

$$\forall_a\, f(a) + f(3a) = 30a^2 - 20a + 14$$

On the other hand, the occurrence of (**) at the end of a test-pattern would indicate that the test-pattern furnished a means of verifying any instance of the generalization obtained by quantifying (**). [Such a test-pattern might consist of a derivation of the open sentence '$f(a) + f(3a) = 30a^2 - 20a + 14$' from premisses, one of which is the open sentence '$f(x) = 3x^2 - 5x + 7$', followed by conditionalizing, and discharging the latter assumption [see, for example, steps (4) - (11) of the proof on page 7-64].

Another correct interpretation, and perhaps a more natural one, is that the completed Sample is a short way of saying:

Let f be the function g such that $\forall_x\ g(x) = 3x^2 - 5x + 7$.

Then, $\forall_a\ f(a) + f(3a) = 30a^2 - 20a + 14$.

Here the first sentence defines the symbol 'f' as [for the purpose of this exercise] the function denoted by the definite description:

the function g such that $\forall_x\ g(x) = 3x^2 - 5x + 7$

The justification for using such a description is that there is a function $\{(x, y): y = 3x^2 - 5x + 7\}$ which conforms to this description, and that there are not two such functions. Thus, in this second interpretation, the symbol 'f' acts [temporarily] as a noun, just as symbols such as 'cos' and 'log' do [permanently], while the symbol 'g', like the symbol 'f' in the first interpretation, is an apparent variable--that is, an index which links an operator [in the case of 'g', 'the function...such that', in the case of 'f', '∀...'] to an argument-place in a sentence [see Unit 2, TC[2-27]o, and the COMMENTARY which precedes it].

As an aid in case you wish to write more exercises like those in Part B, but with variations in format, here are some other ways of restating exercises 1, 2, 4, and 5.

1. If $\forall_x\ f(x) = 4x^2 - 8x - 3$ then $\forall_a\ f(2a) + f(a) =$ _____.

   If $\forall_x\ f(x) = 4x^2 - 8x - 3$ then $\forall_y\ f(2y) + f(y) =$ _____.

   $\forall_b$ if $\forall_x\ f(x) = 4x^2 - 8x - 3$ then $f(2b) + f(b) =$ _____.

   If $\forall_x\ f(x) = 4x^2 - 8x - 3$ then $f(2b) + f(b) =$ _____.

   If $f(x) = 4x^2 - 8x - 3$, $\forall_a\ f(2a) + f(a) =$ _____.

   $f(x) = 4x^2 - 8x - 3$; $f(2a) + f(a) =$ _____.

   Suppose that $\forall_x\ f(x) = 4x^2 - 8x - 3$. Then, $f(2a) + f(a) =$ _____.

   Suppose that $\forall_x\ f(x) = 4x^2 - 8x - 3$. Then, $\forall_a\ f(2a) + f(a) =$ _____.

   What is $f(2a) + f(a)$ if $f(x) = 4x^2 - 8x - 3$?

2. If $\forall_x\ g(x) = 2x^3 - x^2 + 7$ then $\forall_x\ g(3x) - g(-x) =$ ____ .
If $\forall_x\ g(x) = 2x^3 - x^2 + 7$ then $\forall_z\ g(3z) - g(-z) =$ ____ .
[But not: $\forall_x$ if $g(x) = 2x^3 - x^2 + 7$ then $g(3x) - g(-x) =$ ____ .
nor: $\forall_x$ if $g(x) = 2x^3 - x^2 + 7$ then $g(3z) - g(-z) =$ ____ .
nor: $\forall_z \forall_x$ if $g(x) = 2x^3 - x^2 + 7$ then $g(3z) - g(-z) =$ ____ .]

[The notation '[3g]' in Exercise 3 is explained on page 5-102 in Unit 5. '[3 ∘ g]' is explained on page 5-72.]

4. If $\forall_x\ f(x) = 7x^2 + 2x + 9$ then $f(3) =$ ____ and $f(10) =$ ____ .
[But not: $\forall_x$ if $f(x) = 7x^2 + 2x + 9$ then $f(3) =$ ____ and $f(10) =$ ____ .]

5. If $\forall_x\ (f(x) = 8x^3 + 7x^2 + 5x + 6$ and $g(x) = 4x^2 + 3x + 9)$,
$f(10) + g(10) =$ ____ .
If $(\forall_x\ f(x) = 8x^3 + 7x^2 + 5x + 6$ and $\forall_x\ g(x) = 4x^2 + 3x + 9)$,
$f(10) + g(10) =$ ____ .

*

Answers for Part B.

1. $20a^2 - 24a - 6$
2. $56x^3 - 8x^2$
3. $27x^3 - 9x^2 + 7$; $3x^3 - 3x^2 + 21$; $3x^3 - 3x^2 + 21$; 3
4. 78; 729
5. 9195
6. $2 + h$

*

Answers for Part C.

[Identity]; [(1)]; [definition]; [(2), (3)]; [theorem]; [(5)]; [(4), (6)]; [definition]; [(7), (8)]; [definition]; [(9), (10)]

As is the case with addition, the multiplication table and multiplication algorithm are first learned in application to cardinal numbers. That they apply equally well to positive real integers is due to the isomorphism with respect to multiplication between the nonzero finite cardinals and the positive real integers. For finite cardinal numbers and, so, for positive real integers, multiplication can be thought of as repeated addition--in fact, multiplication is, in these cases, sometimes defined to be repeated addition. That this is possible is clear from the fact that, in constructing the multiplication table we need use only the two theorems:

$$\forall_x \, x \cdot 1 = x$$

$$\forall_x \forall_y \, x(y + 1) = xy + x$$

which, when the domain of 'y' is restricted to be the set of positive integers, do characterize multiplication by a positive integer as repeated addition. [We did, in fact, use other theorems in constructing the multiplication table, but they were needed only to get into, and out of, the decimal notation.]

Note, however, that multiplication of real numbers [or of numbers of arithmetic] can <u>not</u>, in general, be thought of as repeated addition. One cannot multiply $\sqrt{3}$ by $\sqrt{2}$ by starting with $\sqrt{3}$ and repeating, $\sqrt{2} - 1$ times, the operation adding $\sqrt{3}$. Nor can one effect the result of multiplying by $\frac{1}{2}$ by repeating some operation $\frac{1}{2} - 1$ times. [Multiplication of real numbers can be defined, as in Unit 2, pages 2-28 and 2-29, in terms of multiplication of numbers of arithmetic. And the latter can be defined, in terms of addition and multiplication of cardinal numbers, by a procedure which involves consideration of sets of sets of ordered pairs of cardinal numbers and a limiting process.]

And, here is an outline of the justification of the algorithm as applied to simplifying '38 • 78':

$$
\begin{array}{ll}
\ 38 \cdot 78 & \\
 & \}\ \text{definition} \\
= 38(7 \cdot 10 + 8) & \\
 & \}\ \ell\text{dpma, apm} \\
= (38 \cdot 7) \cdot 10 + 38 \cdot 8 & \\
 & \}\ \text{multiplication algorithm} \\
= 266 \cdot 10 + 304 & \\
 & \}\ \text{definition} \\
= (2 \cdot 10^2 + 6 \cdot 10 + 6) \cdot 10 + 304 & \\
 & \}\ \text{dpma, apm} \\
= 2 \cdot 10^3 + 6 \cdot 10^2 + 6 \cdot 10 + 304 & \\
 & \}\ \text{definition} \\
= 266\underline{0} + 304 & \\
 & \}\ \text{addition algorithm} \\
= 2964 &
\end{array}
$$

Of course, in the application of the algorithm, the underlined '0' is usually not written, its place being left blank:

$$
\begin{array}{r}
38 \\
\underline{78} \\
304 \\
\underline{266} \\
2964
\end{array}
$$

Students should be able to explain the shift by referring to this missing '0'. They may be interested in a similar phenomenon in an algorithm for long column-additions:

$$
\begin{array}{r}
12 \\
2 \\
75 \\
19 \\
37 \\
\underline{46} \\
31 \\
\underline{16} \\
191
\end{array}
$$

Incidentally, this algorithm is easily adapted to row-addition:

$$12 + 2 + 75 + 19 + 37 + 46 = 31 + 160 = 191$$

In filling out the 2-row in the multiplication table one would prove '$2 \cdot 2 = 4$' before proving '$2 \cdot 3 = 6$'. In these circumstances, the proof of the latter theorem would go like this:

$$2 \cdot 3 \underset{\substack{\uparrow \\ \text{definition}}}{=} 2(2+1) \underset{\substack{\uparrow \\ \ell\text{dpma}}}{=} 2 \cdot 2 + 2 \cdot 1 \underset{\substack{\uparrow \\ \text{pm1}}}{=} 2 \cdot 2 + 2 \underset{\substack{\uparrow \\ \text{theorem}}}{=} 4 + 2 \underset{\substack{\uparrow \\ \text{theorem}}}{=} 6$$

As indicated later on page 7-7, it is convenient to combine the second and third steps into one step which can be justified by the theorem:

$$\forall_x \forall_y\, x(y + 1) = xy + x$$

[This theorem is a consequence of the basic principles cpm, dpma, and pm1.]

*

$2 \cdot 5 = 2(4 + 1) = 2 \cdot 4 + 2 = 8 + 2 = 10;$

$2 \cdot 6 = 2(5 + 1) = 2 \cdot 5 + 2 = 10 + 2 = 1 \cdot 10 + 2 = 12;$

$2 \cdot 7 = 2(6 + 1) = 2 \cdot 6 + 2 = 12 + 2 = (1 \cdot 10 + 2) + 2 = 1 \cdot 10 + (2 + 2)$
$\quad = 1 \cdot 10 + 4 = 14;$ Etc.

*

The grade-school multiplication algorithm can be justified in, say, the case of simplifying '$38 \cdot 8$' as follows:

$$\begin{aligned}
& 38 \cdot 8 && \text{definition} \\
&= (3 \cdot 10 + 8) \cdot 8 && \text{dpma} \\
&= (3 \cdot 10) \cdot 8 + 8 \cdot 8 && \text{multiplication table} \\
&= (3 \cdot 10) \cdot 8 + 64 && \text{definition} \\
&= (3 \cdot 10) \cdot 8 + (6 \cdot 10 + 4) && \text{apm, cpm, apa, dpma} \\
&= (3 \cdot 8 + 6) \cdot 10 + 4 && \text{multiplication table} \\
&= (24 + 6) \cdot 10 + 4 && \text{addition algorithm} \\
&= 30 \cdot 10 + 4 && \text{definition} \\
&= (3 \cdot 10 + 0) \cdot 10 + 4 && \text{dpma, apm, } 10 \cdot 10 = 10^2 \\
&= 3 \cdot 10^2 + 0 \cdot 10 + 4 && \text{definition} \\
&= 304
\end{aligned}$$

Answers for Part A.

4; 3

*

Like the addition algorithm, the multiplication algorithm provides a rather pedestrian procedure for simplifying expressions. An obvious simplification consists in refraining from copying the given expressions. Thus, a solution for Example 1 on page 7-8 might look like this:

$$\begin{array}{r} 9x^2 + 6x + 3 \\ \underline{21x^3 + 14x^2 + 7x \phantom{{}+3}} \\ 21x^3 + 23x^2 + 13x + 3 \end{array}$$

A more economical method consists in merely writing the answer at sight, as illustrated earlier in discussing the addition algorithm. The procedure is more easily carried out than described.

*

Answers for Part B [on pages 7-8 and 7-9].

[Students should discover the advantage in transforming the given expression so that exponents are arranged in ascending or descending order.]

1. $10t^3 - 9t^2 + 16t + 15$
2. $-7x^3 + 46x^2 + 26x - 35$
3. $40y^4 + 8y^3 - 23y^2 - 3y + 3$
4. $8a^4 - 2a^3 + 11a^2 + 38a - 7$
5. $24b^4 + 30b^3 - 12b^2 + 13b + 35$
6. $21r^5 + 6r^4 - 11r^3 - 4r^2 - 2r$
7. $-14k^5 - 46k^3 + 21k^2 + 40k - 15$
8. $22s^4 - 2s^3 + 27s^2 - 3s - 9$
9. $35 + 25x - 14x^2 - 10x^3 + 21x^4 + 15x^5$
10. $32y^5 + 20y^4 + 9y^3 + 23y^2 - 6y$

11. $2x^5 - 8x^4 - 3x^3 + 12x^2 + x - 4$

12. $6y^4 - 8y^3 - 3y^2 + 3y + 1$

13. $-a^5 + 3a^3 - 2a^2$

14. $\frac{1}{3}a^7 - \frac{41}{72}a^6 + \frac{5}{6}a^5 + \frac{1}{4}a^4 - \frac{4}{3}a^3 + \frac{5}{4}a^2 - 1$

15. $10(q-1)^5 - 19(q-1)^4 + 55(q-1)^3 - 51(q-1)^2 + 73(q-1) - 56$

16. $-63x^5 + 139x^4 - 49x^3 - 30x^2$

*

Answers for Part C.

[Students should <u>not</u> use the division-with-remainder algorithm for Part C. These exercises are exploratory for the work starting on page 7-12. It will probably be best to precede that page with more exercises like those in Part C.]

1. $3x + 5$ 2. $x^2 + 3$ 3. $5x^2 + 3x + 4$

*

Answers for Part D.

[Identity]; [(1)]; [definition]; [(2), (3)]; [theorem]; [(5)]; [(4), (6)]; [theorem]; [(7), (8)]; [theorem]; [(9), (10)]

This is the first of several sets of MISCELLANEOUS EXERCISES. The purpose of these exercises is to maintain and to develop further the skills which students have already acquired. At the beginning of the COMMENTARY for each set the exercises will be classified as easy, medium, or hard.

[easy: A 1, B1, 2, C 1, D 1-5, E1, 2, G 2-4; medium: A 2, B 3-6, C 2, D 6, G 1; hard: A 3, B 7-10 [tricks], C 3, F 2, G 5, 6]

*

Answers for Part A.

1. $6x^4 - x^3 - 16x^2 + 46x - 35$

2. $15x^3 - 35x^2 - 3x + 7$; [same]

3. $16x^2 + 20x + 3$; $8x^2 + 4x - 5$

*

Answers for Part B.

1. $2x^2 + x - 41$
2. $2y^2 - 10y + 20$
3. $8y^2 + 13y - 67$
4. $38a^2 + 11ab - 5b^2$
5. $-12x^2 + 4xy - 6y^2$
6. $x^2 - 132x + 71$
7. $10y^2 - 14y$
8. $2x^2 + 25xy + 33y^2$
9. $12x^2 - 7xy - 12y^2$
10. 0

*

Answers for Part C.

1. 60
2. 28
3. 13; $\sqrt{130}$

*

Answers for Part D.

[Students should first multiply factors in pairs (1st · 2nd, 3rd · 4th) and then multiply these products.]

1. $x^4 - 12x^3 + 33x^2 + 38x - 168$
2. $y^4 + 9y^3 - 16y^2 - 156y + 288$
3. $a^4 + 10a^3 + 35a^2 + 50a + 24$
4. $b^4 - 13b^2 + 36$
5. $a^4 + 2a^3 - 113a^2 - 114a + 3024$
6. $120x^4 - 202x^3 - 377x^2 + 518x - 147$

*

Answers for Part E.

1. 8; -32; $49/8$; 4; 8; $521/64$; 7.887875

2. 5863; 3531; 30691

*

Answers for Part F.

1. 142857 [The solution is facilitated by writing down the 3-times table. Then, the argument goes as follows:

   $3E = \_1$ so, recalling the 3-times table, $3E = 21$ and $E = 7$.

   Since $3E = 21$, $3D + 2 = \_E$. So, $3D = \_5$. Hence, $3D = 15$ and $D = 5$.

   Since $3D = 15$, $3C + 1 = \_D$. So, $3C = \_4$. Hence, $3C = 24$ and $C = 8$.

   Since $3C = 24$, $3B + 2 = \_C$. So, $3B = \_6$. Hence, $3B = 06$ and $B = 2$.

   Since $3B = 06$, $3A + 0 = \_B$. So, $3A = \_2$. Hence, $3A = 12$ and $A = 4$.

   Since $3A = 12$, $3 \cdot 1 + 1 = A$. But, $A = 4$. Check.]

2. There are 145 solutions with $M \neq 0$, but only one:

$$\begin{array}{r} 9567 \\ 1085 \\ \hline 10652 \end{array}$$

   in which different letters represent different digits. Here is an analysis of the problem:

Since the sum of any two four-digit numbers is less than 20000, it is clear [assuming that $M \neq 0$] that $M = 1$. Since, in any addition of two numbers, 1 is the only number which is "carried", S is 8 or 9, according as there is carrying to the left-hand column or not. In either case, since $M = 1$, $O = 0$. But, since $O = 0$, nothing is carried to the left-hand column and $S = 9$. Looking now at the column next to the left-hand one, it is clear, since $O = 0$, that, if there is no carrying then $E = N$, while if there is carrying then $1 + E = N$. In the first of these two cases $R = 0$ and $D + E = Y$. E can be any digit-number, and D can be any digit-number such that $D \leq 9 - E$. There are 55 such solutions, in each of which different letters represent the same digit.

In case $1 + E = N$ there are two sub-cases: $D + E = Y$ and $D + E = 10 + Y$. In the first of these $N + R = 10 + E$ and in the second $1 + N + R = 10 + E$.

Since $1 + E = N$, it follows that, in the first sub-case, $R = 9$, and in the second $R = 8$. Since $S = 9$, the first sub-case gives no solutions in which different letters represent different digits. However, waiving this restriction, E can represent any digit except 9 [$E = N - 1$] and, much as in the first main case, there are 54 solutions.

Finally, if $1 + E = N$, $D + E = 10 + Y$, and $R = 8$, then there are 36 solutions. In fact, E may be any digit-number other than 0 or 9, and D may be any digit-number such that $D \geq 10 - E$. However, if different letters are to represent different digits then E cannot be 1 [M] or 8[R], and it cannot be 7 since, in this case, N would be 8. Also, D cannot be 8, 9[S], E, or $1 + E$; and $D + E$ cannot be 10 or 11. These restrictions are sufficient to rule out 35 of the 36 solutions, leaving only that in which $E = 5$, $N = 6$, $D = 7$, and $Y = 2$.

There are more solutions if 0 is allowed as a value for 'M'.

*

Answers for Part G.

1. 4 to 3
2. 11
3. 66, 68, 70
4. 700
5. 666 [1 digit for each of the first 9 pages, 2 for each of the next 90 pages, and 3 for each of the remaining 567 pages--$1890 - (1 \cdot 9 + 2 \cdot 90) = 1701 = 3 \cdot 567$.]

   [A second method of solution is to note that if 3-digit numerals:

   $$001,\ 002,\ \ldots,\ 099,\ 100,\ \ldots$$

   were used the printer would require $2 \cdot 9 + 1 \cdot 90$ extra zeros--1890 + 108 digits in all. So, the number of pages is 1998/3.]

6. If the units-digit of N^2 is a '1' then the units-digit of N is either a '1' or a '9'. Hence, if, for some nonnegative integer m, $100m + 11$ is a perfect square then it is either $(10n + 1)^2$ or $(10n - 1)^2$, for some nonnegative integer n. For short, there is a nonnegative integer n such that

   $$100m + 11 = (10n \pm 1)^2$$

   --that is, such that

   $$10m + 1 = 10n^2 \pm 2n$$

Now, since 100m is divisible by 4 and 10 is not, $100m + 10$ is not divisible by 4. But, since $100n^2$ and $20n$ are both divisible by 4, so are both $100n^2 + 20n$ and $100n^2 - 20n$. Consequently, there are not nonnegative integers m and n such that $100m + 11 = (10n \pm 1)^2$--no decimal numeral for a perfect square ends in '11'.

[A second proof depends on showing that if the units-digit of a positive integer N is a '1' or a '9' then the tens-digit-number of N^2 is even. This can be proved by finding the decimal numerals for the squares of 1, 11, 21, ..., 91, and those of 9, 19, 29, ..., 99. Another method is to note that, in general,

$$(\ldots + n_1 \cdot 10 + n_0)^2 = \ldots + (2n_0 n_1) \cdot 10 + n_0{}^2.$$

Now, if n_0 is neither 4 nor 6, either there will be no carrying to the tens-column or an even number will be carried to the tens-column. Since $2n_0n_1$ is even, so will be the sum of $2n_0n_1$ and what is carried to the tens-column. Since carrying to the hundreds-column reduces the sum by a multiple of 10 (and since 10 is even), the tens-digit-number of N^2 is even (if the units digit of N is neither a '4' nor a '6'). This analysis suggests the following generalization of Exercise 6: The tens-digit-number of a square is odd if and only if its unit-digit is a '4' or a '6'.]

Just as multiplication of real numbers agrees only in exceptional cases with repeated addition, so, division of real numbers agrees only in exceptional cases with repeated subtraction. Nevertheless, the repeated-subtraction algorithm [or: the division-with-remainder algorithm] is an important one. It provides a way of finding the decimal numeral for the quotient of one positive integer by another in cases when this quotient is an integer. In general, it provides a way of calculating the "integral part" of the quotient of two positive integers, as well as its "fractional part". [For the terms 'integral part' and 'fractional part', see pages 7-102 and 7-107.] The algorithm should be thought of as a procedure for obtaining theorems like that at the bottom of page 7-13. It is theorems of this kind which the algorithm actually provides. The corresponding weaker theorems, like:

$$\forall_{x \neq 3} \ \frac{5x^4 + 2x^2 + 3x + 5}{x - 3} = 5x^3 + 15x^2 + 47x + 144 + \frac{437}{x - 3},$$

can be obtained at once from the former whenever they are needed.

The examples on page 7-13 should be demonstrated step by step.

*

Answers for Part A.

1. 25; 4
2. 179; 20
3. $2x^2 - 3x + 7$; 2
4. $5x^3 - x + 4$; 0
5. $3x^2 + 3x + 4$; 5
6. $3x^2 - 14x + 43$; $-160x + 217$
7. $x^2 + 5$; 0
8. $9x^2 - 12x + 16$; 0
9. $x^2 - 4x + 4$; 0
10. $2x^3 + 10x^2 + 50x + 247$; 1242
11. $2x^3 - 8x^2 + 32x - 131$; 531

* * *

Substitute '3' for 'x' in (*). Since 3 - 3 = 0, it follows from the pm0 that

$$5 \cdot 3^4 + 2 \cdot 3^2 + 3 \cdot 3 + 5 = 437$$

Answers for Part B.

1. $f(2) = 29$, $f(6) = 7573$, $f(-7) = 13007$, $f(2/3) = -11$

2. (a) $g(2) = g(3) = g(4) = g(5) = 0$
   (b) The solution set is $\{2, 3, 4, 5\}$.

Note that the results of 2(a) show only that 2, 3, 4, and 5 are roots of the given equation. It requires an additional argument to show that these are the only roots. One procedure is to use the division-with-remainder algorithm to show, successively, that, for each x,

$$\begin{aligned} x^4 - 14x^3 + 71x^2 - 154x + 120 &= (x^3 - 12x^2 + 47x - 60)(x - 2) \\ &= (x^2 - 9x + 20)(x - 3)(x - 2) \\ &= (x - 5)(x - 4)(x - 3)(x - 2). \end{aligned}$$

Then, by the 0-product theorem, used three times, it follows that 2, 3, 4, and 5 are the only roots.

3. 0; $5^3 - 9 \cdot 5^2 + 23 \cdot 5 - 15 = 0$

4. Use the division-with-remainder algorithm, or show that the factors '$(x + 2)$' and '$(x - 1)$' are factors of '$x^5 - x^4 + 9x^2 - 7x - 2$' by the technique used in Exercise 3.

5. Use the technique of Exercise 3. (a), (b), (c), (e), (f)

*

Answers for Part C.

1. $54x^2 - 72x + 24$; $27x^3 - 54x^2 + 36x - 8$

2. $x^3 + 12x^2 + 48x + 64$ cubic inches

3. $2x^2 - 5x + 7$ square feet

4. $(x - 5)\sqrt{2}$ inches; $(x - 5)\sqrt{3}$ inches

The principle of quotients [(4) of the column proof on page 7-16] contains a restricted quantifier, '$\forall_{y \neq 0}$'. The purpose of the restriction '$y \neq 0$' is to rule out meaningless "instances" of the generalization. Consequently, in inferring step (6) one needs to provide some notice that (6) is, indeed, meaningful. We have chosen to do so here by introducing '$6 \neq 0$' as step (5) in the proof. As pointed out later in the COMMENTARY [pages 7-37 through 7-40], a more adequate procedure in cases such as this is to write '$6 \neq 0$' as a restriction on step (6), rather than as an extra step in the proof. Doing so, step (5) would be omitted and step (6) would be:

(6)	$\frac{5}{6} \cdot 6 = 5$	$[6 \neq 0]$	[(4)]

[In fact, '$[6 \neq 0]$' should have appeared as a restriction on step (2).] Similarly, rather than introduce step (9), it would be more appropriate to include the restriction '$[6 \neq 0,\ 3 \neq 0]$' in step (10).

In Unit 2, sentences like (5) and (9) were treated, on a par with sentences like (3), as statements of computing facts. Since we have now seen that, after the introduction of appropriate definitions, (3) is a theorem, it is natural to ask whether (5) and (9) are also theorems. In answering this question we shall discover the need for additional basic principles.

Students have [and should have] no doubt that the real number 1 is different from the real number 0. However, they should by now be ready to wonder whether '$1 \neq 0$' is a theorem--that is, whether the information about the real numbers which is expressed in the ten basic principles and their consequences includes the fact that 1 and 0 are different. [Analogously, students had no doubt that if the sum of two numbers is 0 then one is the opposite of the other. Still, in the deductive development of the theory of the real numbers, it was important to prove the 0-sum theorem.] Now, there is an easy way to see that '$1 \neq 0$' is not a consequence of the ten basic principles of Unit 2. The method is based on two facts. First, whether a sentence is a consequence of the basic principles depends just on the "shape" of the sentence, and not at all on the "meaning" of the sentence. For example, one need not know that the basic principles are generalizations about real numbers in order to derive the 0-sum theorem from the apa, the po, and the pa0 [TC[7-134]c].

As a matter of fact, many people have done just this while interpreting the sentences in question as statements about a quite different subject--or not interpreting them at all. Second, statements which are consequences of true statements are also true. In particular, if we choose any set of things for the domain of the variables 'x', 'y', and 'z', and assign meanings to '0', '1', '+', '•', '−', '-', and '÷' in such a way that in this interpretation, the basic principles are true statements, then each theorem will also be a true statement.

So, to show that '$1 \neq 0$' is not a theorem all we need do is find an interpretation of the language in which the basic principles are written which is of such a kind that the basic principles are true and '$1 \neq 0$' is false. As shown in the text, this is easy to do. For the domain of the variables, take any set which has just one member, and decide to use both '0' and '1' as names for this one thing. Then, '$0 \neq 1$' is certainly false. Since $0 = 1$, it follows that '$0 + 1$', '$1 + 0$', and '$1 + 1$' must all mean the same as '$0 + 0$'. So, to define addition all you need decide is what '$0 + 0$' names. Of course, you have no choice--$0 + 0 = 0$. Similarly, to define multiplication, opposition, and subtraction, you can only decide that $0 \cdot 0 = 0$, $-0 = 0$, and $0 - 0 = 0$. Finally, the only basic principle in which '÷' occurs is the principle of quotients. Since, because of the restricted quantifier, no instance of this principle contains '$0 \div 0$', it doesn't matter which you decide about this symbol. You can, as usual, decide that it shall be meaningless, or you can decide that $0 \div 0 = 0$. In either case, the pq will be a true statement [no counter-example]. It is now easy to check that each of the other nine basic principles is true.

Since there is an interpretation under which the ten basic principles are true and '$1 \neq 0$' is false, it follows that '$1 \neq 0$ is not a consequence of the basic principles. Consequently, the ten basic principles do not formulate all we already know about the real numbers. To do so, more basic principles are needed. In particular, we shall add '$1 \neq 0$' as an eleventh basic principle.

The sentence '$1 \neq 0$' was used, in Unit 2, in the proof of Theorem 50 ['$\forall_x\ x/1 = x$']. So, adopting this sentence as a basic principle fills up a gap in the deductive development in Unit 2. There is just one other gap --the use of '$-1 \neq 0$' in the proof of Theorem 52 ['$\forall_x\ x/-1 = -x$']. As indicated on page 7-18, this gap can now also be filled. For, combining Theorem 79, which is a consequence of our ten original basic principles, with '$1 \neq 0$', it is easy to prove '$-1 \neq 0$':

(1)	$\forall_x$ if $x \neq 0$ then $-x \neq 0$	[Theorem 79]
(2)	if $1 \neq 0$ then $-1 \neq 0$	[(1)]
(3)	$1 \neq 0$	[basic principle]
(4)	$-1 \neq 0$	[(2) and (3)]

*

Theorem 80 ['$-0 = 0$'] can be proved in several ways. The simplest makes use of the 0-sum theorem.

(1)	$\forall_x \forall_y$ if $x + y = 0$ then $-x = y$	[Theorem 16]
(2)	if $0 + 0 = 0$ then $-0 = 0$	[(1)]
(3)	$\forall_x\ x + 0 = x$	[pa0]
(4)	$0 + 0 = 0$	[(3)]
(5)	$-0 = 0$	[(2) and (4)]

Another proof of Theorem 80 uses Theorem 79 and Theorem 17.

(1)	$\forall_x$ if $x \neq 0$ then $-x \neq 0$	[Theorem 79]
(2)	if $-0 \neq 0$ then $--0 \neq 0$	[(1)]
(3)	if $--0 = 0$ then $-0 = 0$	[(2)]
(4)	$\forall_x\ --x = x$	[Theorem 17]
(5)	$--0 = 0$	[(4)]
(6)	$-0 = 0$	[(3) and (5)]

Here are some other pairs of sentences which can be proved equivalent in this way:

$-0 = 0$	$\forall_x$ if $x = 0$ then $-0 = x$
$-0 = 0$	$\forall_x$ if $x = 0$ then $-x = x$
$1 \neq 0$	$\forall_x$ if $x = 1$ then $x \neq 0$
$1 \neq 0$	$\forall_x$ if $x = 0$ then $1 \neq x$
$a \not\geq a$	$\forall_x$ if $x = a$ then $x \not\geq a$

A slight modification of the proof that each of the last two sentences can be derived from the other shows that:

$$\forall_x\, x \not\geq x \qquad \text{and:} \qquad \forall_x \forall_y \text{ if } x = y \text{ then } x \not\geq y$$

are equivalent.

*

It turns out that, even after we have added '$1 \neq 0$' to our basic principles, '$2 \neq 0$' is not a theorem. That this is the case can be shown by finding an interpretation of the language of our eleven basic principles which is such that the basic principles become true statements and '$1 + 1 \neq 0$' is false. [See the COMMENTARY for pages 7-16 and 7-17.] Such an interpretation can be made by choosing a set consisting of just two things for the domain of 'x', 'y', and 'z', deciding to call one of these things '0', and the other '1', and defining the five operations by the following tables:

+	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

−	
0	0
1	1

-	0	1
0	0	1
1	1	0

÷	0	1
0		0
1		1

[The two blanks left in the '÷' table can either be left blank (division by 0 undefined) or, either or both, filled with a '0' or a '1'.] It is easy to check that the eleven basic principles are true in this interpretation, and that '$1 + 1 \neq 0$' is false. So, the latter statement is not a consequence of the eleven basic principles.

The proof that:

$$\text{(a)} \quad -0 = 0$$

is equivalent to:

$$\text{(b)} \quad \forall_x \text{ if } x = 0 \text{ then } -x = 0$$

illustrates a trick of logic which students will have several occasions to use. From (a) one can derive (b) as follows:

(1)	$-0 = 0$		$F(s)$
(2)	$a = 0$	[assumption]*	$a = s$
(3)	$-a = 0$	[(1) and (2)]	$F(a)$
(4)	if $a = 0$ then $-a = 0$	[(3); *(2)]	if $a = s$ then $F(a)$
(5)	$\forall_x$ if $x = 0$ then $-x = 0$	[(1) - (4)]	$\forall_x$ if $x = s$ then $F(x)$

And, from (b) one can derive (a) as follows:

(1)	$\forall_x$ if $x = 0$ then $-x = 0$		$\forall_x$ if $x = s$ then $F(x)$
(2)	if $0 = 0$ then $-0 = 0$	[(1)]	if $s = s$ then $F(s)$
(3)	$\forall_x \; x = x$	[Identity]	$\forall_x \; x = x$
(4)	$0 = 0$	[(3)]	$s = s$
(5)	$-0 = 0$	[(2) and (4)]	$F(s)$

In general, if, as suggested in the right-hand column, we think of '$F(s)$' as an abbreviation of some sentence containing an expression which we abbreviate by 's', and think of '$F(x)$' as an abbreviation of the sentence obtained from the first by replacing the expression in question by an 'x', then the same sort of arguments show that:

$$\text{(a}'\text{)} \quad F(s) \quad \text{is equivalent to:} \quad \text{(b}'\text{)} \quad \forall_x \text{ if } x = s \text{ then } F(x)$$

Answers for Miscellaneous Exercises.

[You should, in making assignments, take care not to break up short sequences of related exercises such as Exercises 10 and 11 of Part A, and Exercises 13, 14, and 15, and 17, 18, and 19 of Part C.]

[<u>easy</u>: A1-8, 12-15, B1-6, C1-7, 10, 11, 13-15, 22, 23, 25, 26, D1-5, E1, 2, 4-6; <u>medium</u>: A9-11, C8, 9, 12, 16-18, 20, 24, 26, D6, E3, 7; <u>hard</u>: C19, 21]

*

Answers for Part A.

1. $25a^2 - b^2$
2. $x^2 + 15x + 56$
3. $y^2 + 18y + 81$
4. $y^2 - 81$
5. $9x^2 - 24x + 16$
6. $9z^2 - 16$
7. $9z^2 + 24z + 16$
8. $35a^2 - 32a - 12$
9. $a^2x^2 + abx - acx - bc$
10. $a^2 - 2ab + b^2 - c^2$
11. $x^2 - 2xy + y^2 - z^2$
12. $5p^3q - 5p^2q^2 - 5pq^3$
13. $7x^6y^3 - 7x^4y^4 + 7x^2y^7$
14. $10a^3 - 19a^2b + 20ab^2 - 21b^3$
15. $15x^4 - 31x^3 + 13x^2 - 3x - 2$

*

Answers for Part B.

1. 8100
2. 6700
3. 1900
4. 6300
5. 1200
6. 5500

*

Answers for Part C.

1. $\frac{x^2 - x + 16}{x(x + 8)}$

2. $\frac{x^2 - x + 15}{x(x - 3)}$

3. $\frac{16 + 9y - y^2}{y(y + 2)}$

4. $\frac{10x - 13}{(x - 4)(x + 5)}$

5. $\frac{2x^2 - 3x - 17}{(x + 4)(x - 5)}$

6. $\frac{2x}{(x + 4)(x + 5)}$

7. $\frac{2x^2 - 6x + 50}{(x - 5)(x + 5)}$

8. $\frac{25x^2 - 49x - 4}{(5x + 1)(5x - 1)}$

9. $\frac{24y + 7}{3(y - 2)}$

10. $\frac{35a^2x}{24by}$

11. $\frac{m^2q}{pn}$

12. $\frac{4(a^2 - b^2)}{3}$

13. $\frac{(x - 3)(x - 7)}{(x + 4)(x - 5)}$

14. $\frac{x - 7}{x + 4}$

15. $\frac{(x - 4)(x - 3)}{(x - 2)(x + 4)}$

16. $\frac{2x - 1}{2x - 3}$

17. $\frac{3x^2 - 8x + 5}{(x - 3)(x - 2)}$

18. $\frac{4x^2 - 15x + 6}{(x - 3)(x - 2)}$

[illegible]

19. $\frac{x^3 - 51x^2 - 46x + 480}{(x + 8)(x - 3)(x - 8)}$

20. $\frac{2x^2 - 23x - 48}{3(x - 1)(x - 9)}$

21. $\frac{20 + 25x - 14x^2}{(2x - 3)(x + 5)}$

22. $\frac{2x - 5}{x + 4}$

23. $\frac{y - 4}{2}$

24. $\frac{10a^2 - 3ab - 11b^2}{(a + b)(a - b)}$

25. 1

26. 3

27. $\frac{12x^2 + 25x - 33}{(6 - x)(x - 3)}$

*

Answers for Part D.

1. $\left(\frac{7}{3}, \frac{1}{3}\right)$

2. $\left(\frac{5}{2}, 0\right)$

3. $\left(\frac{1}{2}, \frac{3}{2}\right)$

4. $\left(\frac{5}{2}, 0\right)$

5. [no solution]

6. $\left(-\frac{5}{9}, \frac{8}{9}\right)$

*

Answers for Part E.

1. 11, 12

2. $3\frac{3}{7}$ days

3. $70\frac{5}{7}$ miles per hour

4. 92

5. 9 hours

6. 240 square feet

7. Since $\ell = 2\sqrt{100 - d^2}$, there is a function f such that $\ell = f \circ d$. Such a function is $\{(x, y),\ -10 \le x \le 10 : y = 2\sqrt{100 - x^2}\}$. [The intersection of all such functions is $\{(x, y),\ 0 \le x < 10 : y = 2\sqrt{100 - x^2}\}$.]

In section 7.02 we adopt four new basic principles, (P_1) - (P_4). As was the case with each of the eleven preceding basic principles, each of the new principles formulates something students have previously discovered about the real numbers. [As you have probably guessed, (P_1) - (P_4) formulate some important properties of the set P whose members are the positive numbers.]

*

The request, at the end of the second paragraph, for an explanation, may be fulfilled in either of two ways.

(I) We have noticed that

$$1 + 1 = 0 \text{ if and only if } -1 = 1.$$

In particular [only if-part],

$$\text{if } 1 + 1 = 0 \text{ then } -1 = 1.$$

So, if we can prove that $-1 \neq 1$, we can conclude [by modus tollens] that $1 + 1 \neq 0$--that is, that $2 \neq 0$.

In the second explanation, to be given presently, the substitution rule for biconditional sentences takes the place of modus tollens. The substitution rule for biconditional sentences [Unit 6, page 6-391] may be formulated as follows:

> Given a biconditional sentence and another sentence, if, in the second sentence, either component of the biconditional sentence is replaced by its other component then the new sentence so obtained is a consequence of the given sentences.

(II) We have noticed that

$$(*) \qquad 1 + 1 = 0 \text{ if and only if } -1 = 1.$$

So, if we can prove that

$$\text{not } [-1 = 1]$$

then we can infer [by the substitution principle for biconditional sentences--replacing in our conclusion the right component of (*) by its left component] that

$$\text{not } [1 + 1 = 0]$$

--that is, that $2 \neq 0$.

In giving explanation (II) we have written 'not $[-1 = 1]$' and 'not $[1 + 1 = 0]$' rather than the simpler '$-1 \neq 1$' and '$1 + 1 \neq 0$'. The reason is that, at first, people sometimes find it difficult to see, for example, the '$-1 = 1$' in the sentence '$-1 \neq 1$'. Once arguments using the substitution principle for biconditional sentences are understood, it should be unnecessary to resort to this dodge.

Each of these explanations is logically sound. However, the second illustrates a use of biconditional sentences with which students should become thoroughly familiar. They will have an opportunity to practice this kind of argument in Part D on pages 7-26 and 7-27, and in Parts A and C on pages 7-31 and 7-32, as well as later in this unit. But this is the point at which to make sure that they are aware of it.

The advantage of explanations like (II) shows up more clearly in cases in which the "second sentence" is more complicated. Consider the problem of deriving:

(1) if $a > 0$ and $b > c$ then $ab > ac$

from:

(2) if $a \in P$ and $b > c$ then $ab > ac$

and:

(3) $a > 0$ if and only if $a \in P$

Using the substitution rule for biconditional sentences, the derivation consists of just one inference. Schematically:

$$\frac{(3) \qquad\qquad (2)}{(1)}$$

A derivation following the lines of an explanation like (I) would be quite complicated. In paragraph form [to save space] it would go something like this:

> Suppose that (3) and (2), and suppose that $a > 0$ and $b > c$. It follows that $a > 0$ and so, by the only if-part of (3), that $a \in P$. Since it also follows that $b > c$, $a \in P$ and $b > c$. Hence, by (2), $ab > ac$. Consequently, if $a > 0$ and $b > c$ then $ab > ac$.

An analysis of this argument shows that it uses seven inferences to accomplish what was done by the single inference based on the substitution rule for biconditional sentences.

*

Students should realize that (P_1) and (P_2) [and, later, (P_3) and (P_4)] are not meant to communicate to them new information about the real numbers. Students already know that [(P_1)], given any nonzero real number, either it or its opposite is positive, and that [(P_2)] there is no number such that it and its opposite are both positive. However, this information could not be predicted from the preceding eleven basic principles because these contain nothing which can be used to distinguish between positive and nonpositive numbers. Our purpose in choosing these few new basic principles is to call attention to a small number of simple properties of the positive numbers, knowledge of which, together with the preceding eleven basic principles, enables one to deduce all properties of these numbers [more precisely: all properties which depend essentially on "positiveness"]. As in the past, basic principles help us to organize our present knowledge and to discover new results.

*

In discussing (P_1) and (P_2), emphasize the word 'opposite'. Here is a place to stress the difference between 'opposite' and 'negative' [which, in reading (P_1) and (P_2) would be incorrect] and between 'opposite' and 'minus' [which is, at <u>best</u>, neutral]. One way to stress the difference between 'opposite' and 'negative' is to call attention to such instances of (P_1) as:

$$^{-}2 \neq 0 \Longrightarrow \text{ either } {}^{-}2 \in P \text{ or } -{}^{-}2 \in P,$$

reading this as 'if negative 2 is not 0 then either negative 2 belongs to P or the opposite of negative 2 belongs to P'. Also, consider similar instances of (P_2).

There are theorems analogous to (P_1) and (P_2) which deal explicitly with negative numbers:

$$\forall_x \, [x \neq 0 \Longrightarrow \text{ either } x \in P \text{ or } x \in N],$$

[each nonzero number is either positive or negative] and:

$$\forall_x \text{ not both } x \in P \text{ and } x \in N,$$

[no number is both positive and negative]. They are taken up in Part D on pages 7-26 and 7-27.

*

The words 'either' [in (P_1)] and 'both' [in (P_2)] are pillow-words--they add nothing to the meaning of either statement. They do play a role as punctuation--without them we would need parentheses:

$$\forall_x \, [x \neq 0 \Longrightarrow (x \in P \text{ or } -x \in P)]$$

$$\forall_x \text{ not } [x \in P \text{ and } -x \in P]$$

*

The argument summarized in the sentence on line 8 from the bottom of page 7-22 makes use of the rule of the dilemma [see page 6-393 of Unit 6]:

$$\frac{p \text{ or } q \qquad p \Rightarrow r \qquad q \Rightarrow r}{r}$$

In tree-form the argument runs as follows:

$$\dfrac{1 \in P \text{ or } -1 \in P \qquad \dfrac{\dfrac{1 \stackrel{*}{\in} P \qquad \dfrac{-1 \stackrel{\dagger}{=} 1 \qquad 1 \stackrel{*}{\in} P}{-1 \in P}}{1 \in P \text{ and } -1 \in P}}{1 \in P \Rightarrow (1 \in P \text{ and } -1 \in P)}\,* \qquad \dfrac{\dfrac{\dfrac{-1 \stackrel{\dagger}{=} 1 \qquad -1 \stackrel{*}{\in} P}{1 \in P} \qquad -1 \stackrel{*}{\in} P}{1 \in P \text{ and } -1 \in P}}{-1 \in P \Rightarrow (1 \in P \text{ and } -1 \in P)}\,*}{\dfrac{1 \in P \text{ and } -1 \in P}{-1 = 1 \Rightarrow (1 \in P \text{ and } -1 \in P)}\,\dagger}$$

[A similar argument, with 'a's in place of '1's, forms the nucleus of a proof for the theorem '$\forall_x\, [x \neq 0 \Rightarrow -x \neq x]$'.]

*

Although, as has been seen, it is now possible to prove that $2 \neq 0$, it is not yet possible to prove that $3 \neq 0$. The same method of interpretation used on pages 7-17 and 7-19 shows that '$3 \neq 0$' is not yet a theorem. What one does is to consider any set of three things--call them '0', '1', and '2'. Let P be $\{1\}$, and define the four operations by:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

−	
0	0
1	2
2	1

−	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

÷	0	1	2
0		0	0
1		1	2
2		2	1

[As before (see COMMENTARY for page 7-19), the blank spaces in the 0-column of the division table can be left blank, or filled, arbitrarily,

with '0's, '1's and '2's.] Since, by definition, $2 + 1 = 0$, it follows [using the definition according to which '3' is an abbreviation for '2 + 1'] that '$3 \neq 0$' is false for this interpretation. However, one can, by checking instances, show that each of the thirteen basic principles is true. So, '$3 \neq 0$' is not a consequence of these basic principles.

To help pave the way for the argument, on page 7-23, that the proof of Theorem 82 requires a new basic principle, it will be useful to point out a second interpretation. This is just like the 3-element interpretation just given except that, now, P is $\{2\}$. Since, by definition, $2 = -1$, we have one interpretation--the first--in which 1 is positive and -1 is negative, and a second interpretation in which -1 is positive and 1 is negative. This, in itself, shows that Theorem 82 cannot be derived from our present thirteen basic principles. The argument on page 7-23 shows that even the new basic principle (P_3) [P is <u>closed</u> with respect to addition] does not enable us to prove that 1 is positive.

Incidentally, in neither of the 3-element interpretations of our thirteen basic principles is P closed with respect to addition. So, (P_3) is not a consequence of our thirteen principles.

*

Although you will probably wish to save them for review purposes, note that there is work on alternative interpretations of our basic principles in REVIEW EXERCISES 1 and 2, on pages 7-133 and 7-134.

*

To derive '$3 \neq 0$' from '$3 \in P$' and '$0 \notin P$' one argues as follows:

> Suppose that $3 = 0$. Since $3 \in P$, it follows that $0 \in P$. So, if $3 = 0$ then $0 \in P$. Hence, since $0 \notin P$, it follows that $3 \neq 0$.

Of course, to be of much help, this argument must be buttressed by proofs of '$3 \in P$' and '$0 \notin P$'.

together with the three statements (N_1), (N_2), and (N_3). But these are the theorems in Exercises 3, 4, and 5 of Part D on page 7-27, and are themselves derivable from our fourteen basic principles and the definition of N given at the top of page 7-26. So, combining the original proof of '$1 \in P$' with this modified proof of '$1 \in N$', we see that '$1 \in P$ and $1 \in N$' is a consequence of our fourteen basic principles together with the definition (N) on page 7-26. But '$1 \in P$ and $1 \in N$' contradicts the theorem of Exercise 2 on page 7-27 which, itself, is a consequence of our fourteen basic principles and (N). So, if we could derive '$1 \in P$' from our fourteen basic principles, then these fourteen together with (N) would be inconsistent. This is not the case since the real numbers satisfy all fifteen of these principles. So, we can't derive '$1 \in P$' from the fourteen.

valid sentences, such as '$\forall_x\ x = x$' or '$2 \in P \Longrightarrow 2 \in P$'. Since we could expand the proof to include proofs of the "previously proved theorems", we may suppose that each undischarged premiss of the proof is either one of our fourteen basic principles or a logically valid sentence.

Now, let's replace, throughout the proof, the letter 'P' by another letter, say 'Q'. All the inferences in the proof will still be valid and each of the premisses of the proof which was a logically valid sentence will still be a logically valid sentence. Any of the first eleven basic principles which was a premiss of the original proof will be unchanged, and a premiss which was (P_1), (P_2), or (P_3) will be replaced by:

(Q_1) $\quad \forall_x [x \neq 0 \Longrightarrow \text{either } x \in Q \text{ or } -x \in Q]$,

(Q_2) $\quad \forall_x \text{ not both } x \in Q \text{ and } -x \in Q$,

or: (Q_3) $\quad \forall_x \forall_y [(x \in Q \text{ and } y \in Q) \Longrightarrow x + y \in Q]$,

respectively. The conclusion of the original proof was '$1 \in P$'. So, the conclusion of the new proof will be '$1 \in Q$'.

What this amounts to is to show that if it follows from our fourteen basic principles that $1 \in P$, then 1 belongs to each set Q of real numbers which satisfies (Q_1), (Q_2), and (Q_3).

Now, let's take 'Q' as a name for the set of negative numbers. If we do, then (Q_1), (Q_2), and (Q_3) are true--given any nonzero real number, either it or its opposite is negative; there is no real number such that it and its opposite are both negative; and the sum of two negative numbers is negative.

So, if we could prove that 1 is positive, we could also prove that 1 is negative. Since we know that the real number 1 is not negative, it follows that [using only our fourteen basic principles] we cannot prove '$1 \in P$'.

There is an interesting modification of the preceding argument which runs as follows:

Suppose we have a proof of '$1 \in P$' on the basis of our fourteen basic principles. If in this proof we replace 'P' everywhere by 'N', we obtain a derivation of '$1 \in N$' from our first eleven basic principles

Proof of Theorem 81 [read '$0 \notin P$' as '0 is not positive']:

> By (P_2), not both $0 \in P$ and $-0 \in P$. So, since $-0 = 0$, it follows that not both $0 \in P$ and $0 \in P$--that is, $0 \notin P$.

*

The last step in the proof above may be justified by the logical prinicple:

(*) $$(0 \in P \text{ and } 0 \in P) \Longleftrightarrow 0 \in P$$

and an application of the substitution rule for biconditional sentences. One substitutes the right-hand component of (*) for its left-hand component in 'not both $0 \in P$ and $0 \in P$'--that is, in:

$$\text{not } (0 \in P \text{ and } 0 \in P)$$

--to obtain:

$$\text{not } (0 \in P)$$

That sentences like (*) are logical principles can be shown, à la Unit 6, as follows:

$$\cfrac{\cfrac{\cfrac{\overset{*}{p} \qquad \overset{*}{p}}{p \text{ and } p}}{p \Longrightarrow [p \text{ and } p]}\,{}^{*} \qquad \cfrac{\cfrac{\overset{\dagger}{p \text{ and } p}}{p}}{[p \text{ and } p] \Longrightarrow p}\,{}^{\dagger}}{[p \text{ and } p] \Longleftrightarrow p}$$

The two top inferences are valid by the two basic rules of reasoning for conjunction sentences. The two middle ones are, of course, examples of conditionalizing [and discharging a premiss]. The last is valid by one of the two basic rules for biconditional sentences. [For all these types of inference, see Unit 6, page 6-396.]

*

The argument given to show that our fourteen basic principles, including (P_1), (P_2), and (P_3), are not sufficient for the proof of Theorem 82 is interesting. Since a similar technique is used in annother connection on page 7-48, it is worth spending some time on here. You might expand on the text somewhat as follows:

> Suppose we had a proof of '$1 \in P$'. Its undischarged premisses would be basic principles, previously proved theorems, and logically

Answers for Part A.

1. By the ps, $a - a = a + -a$, and, by the po, $a + -a = 0$. So, $a - a = 0$. Consequently, $\forall_x\ x - x = 0$.

   Suppose that $a = b$. Since $a - a = 0$ it follows [using the substitution rule for equations] that $a - b = 0$. Hence, if $a = b$ then $a - b = 0$. Consequently, $\forall_x \forall_y\ [x = y \Longrightarrow x - y = 0]$.

[The procedure used to prove (*) of Exercise 1 is of frequent utility. [See the COMMENTARY for page 7-19.] Note that not only is (*) a consequence of '$\forall_x x - x = 0$', but also the latter is a consequence of the former. For, assuming that $\forall_x \forall_y\ [x = y \Longrightarrow x - y = 0]$, it follows that if $a = a$ then $a - a = 0$. So, since $a = a$, it follows that $a - a = 0$. Consequently, $\forall_x\ x - x = 0$.]

2. Suppose that $a - b = 0$. It follows that $a - b + b = 0 + b = b$. Since, by Theorem 32, $a - b + b = a$, it follows that $a = b$. Hence, if $a - b = 0$ then $a = b$. Consequently, $\forall_x \forall_y\ [x - y = 0 \Longrightarrow x = y]$.

*

Answers for Part B.

1. By Theorem 82, $1 \in P$. Hence, by (P_3), $1 + 1 \in P$--that is, $2 \in P$. Since $2 \in P$ and $1 \in P$, it follows from (P_3) that $2 + 1 \in P$--that is, that $3 \in P$. And, since $3 \in P$ and $1 \in P$, it follows from (P_3) that $3 + 1 \in P$--that is, that $4 \in P$.

2. Suppose that $2 = 3$--that is, that $2 = 2 + 1$. It follows that $1 + 2 = 2$ and, by (*) of Exercise 1 of Part A, that $1 + 2 - 2 = 0$. But, by Theorem 30, $1 + 2 - 2 = 1$. Hence, if $2 = 3$ then $1 = 0$. Since $1 \neq 0$, $2 \neq 3$. [There are, of course, numerous other proofs.]

3. Suppose that $\frac{2}{3} = 1$. It follows that $\frac{2}{3} \cdot 3 = 1 \cdot 3 = 3$. Since $3 \neq 0$, it follows from the pq that $\frac{2}{3} \cdot 3 = 2$. Hence, if $\frac{2}{3} = 1$ then $2 = 3$. Since [Exercise 2] $2 \neq 3$, it follows that $\frac{2}{3} \neq 1$.

4. Suppose that $a = -a$. It follows that $a + a = a + -a = 0$ and, so, that $a \cdot 2 = 0$. Since $2 \neq 0$, it follows that $a = 0$. Hence, by Theorem 81, $a \notin P$. Therefore, if $a = -a$ then $a \notin P$. So, if $a \in P$ then $a \neq -a$. Consequently $\forall_x\ [x \in P \Longrightarrow x \neq -x]$. [As with many geometry theorems, students should find it helpful to state the theorem as in the last sentence before trying to prove it.]

5. $1 \in P$, but $1 - 1 = 0$ and $0 \notin P$. [Students may find it helpful to state the universal generalization they are trying to disprove (or the existential generalization they are trying to prove).]

Answers for Part C.

1. By Theorem 32, $a - b + b = a$. Hence, by (P_3), if $a - b \in P$ and $b \in P$ then $a \in P$. Consequently, $\forall_x \forall_y [(x - y \in P \text{ and } y \in P) \Rightarrow x \in P]$.

2. Suppose that $a - b \in P$ and $b \in P$. From the theorem of Exercise 1, it follows that $a \in P$. Since $a \in P$ and $b \in P$, it follows from (P_3) that $a + b \in P$. Since $a - b \in P$ and $a + b \in P$, it follows from (P_4) that $(a - b)(a + b) \in P$. But, $(a - b)(a + b) = a^2 - b^2$. So, $a^2 - b^2 \in P$. Therefore, if $a - b \in P$ and $b \in P$ then $a^2 - b^2 \in P$. Consequently, $\forall_x \ldots$.

   [Using basic principle (G) on page 7-30, the theorem just proved becomes:

   $$\forall_x \forall_y \text{ if } x > y > 0 \text{ then } x^2 > y^2$$

   Sample 2 on page 7-25 and Exercises 3, 4, 5, and 6 are similarly important as leading to the theorems on inequations.]

*

The answer to the 'Why?' in the solution of Sample 2 should refer to the theorem of Exercise 2 of Part A on page 7-24. The question is asked principally to catch students who are still being careless in distinguishing between a statement and its converse. Such students may refer, incorrectly, to (*) of Exercise 1 of Part A.

*

3. Since, by Theorem 33, $-(a - b) = b - a$ and since, by (P_2), not both $a - b \in P$ and $-(a - b) \in P$, it follows that not both $a - b \in P$ and $b - a \in P$. Consequently, $\forall_x \forall_y$ not both $x - y \in P$ and $y - x \in P$.

4. Suppose that $a - b \in P$ and $b - c \in P$. It follows, by (P_3), that $a - b + (b - c) \in P$. So, since $a - b + (b - c) = a - c$, it follows that $a - c \in P$. Hence, if $a - b \in P$ and $b - c \in P$ then $a - c \in P$. Consequently, $\forall_x \forall_y \forall_z \ldots$.

5. Suppose that $a - b \in P$. Since, by Theorem 44, for any c, $a + c - (b + c) = a - b$, it follows that $(a + c) - (b + c) \in P$. Hence, if $a - b \in P$ then $(a + c) - (b + c) \in P$. Consequently, $\forall_x \ldots$.

6. Suppose that $c \in P$ and that $a - b \in P$. It follows, by (P_4), that $(a - b)c \in P$. Since, by Theorem 39, $(a - b)c = ac - bc$, $ac - bc \in P$. Hence, if $c \in P$ and $a - b \in P$ then $ac - bc \in P$. Consequently, $\ldots$.

Since our fifteen basic principles do not mention the negative numbers, we need another basic principle if we are to prove theorems about negative numbers. Since the negative numbers can be characterized as the opposites of the positive numbers, this additional basic principle could be merely a restatement of this characterization:

$$(*) \qquad \forall_x [x \in N \iff \exists_y (y \in P \text{ and } x = -y)]$$

[Recall that '$\exists_y$' is read as 'there is a y such that'.]

Such a generalized biconditional is commonly called a definition of the symbol 'N'. More particularly, (*) is called a contextual definition of 'N', since it shows how to replace contexts in which 'N' occurs ['$a \in N$', '$2 - 5 \in N$', etc.] by contexts which do not contain 'N'. [Exercise 1 of Part D could also serve as a contextual definition of 'N'.] Such definitions function as basic principles. Basic principle (G), introduced on page 7-30, is another example of such a definition, as is also basic principle (I), on page 7-94. The principle for subtraction:

$$\forall_x \forall_y\ x - y = x + -y$$

is another example of a basic principle which is, more specifically, a contextual definition. That the ps could be stated as a generalized equation rather than as a generalized biconditional is due to the fact that the symbol '-' names an operation. [In place of the ps we could have used '$\forall_x \forall_y \forall_z [z = x - y \iff z = x + -y]$'. The possible contextual definitions of 'N', (*) and Exercise 1 of Part D, can be transformed into explicit definitions:

$$N = \{x: \exists_y (y \in P \text{ and } x = -y)\}, \quad \text{and:} \quad N = \{x: -x \in P\}$$

[There is an interesting chapter on definition in Introduction to Logic, by P. Suppes (Van Nostrand).]

Referring to (*), it is not difficult to see that, by virtue of Theorem 17 ['$\forall_x\ --x = x$'], (*) is equivalent to the theorem of Exercise 1 of Part D:

$$(**) \qquad \forall_x [x \in N \iff -x \in P]$$

And, as the solution given below for Exercise 1 shows, this theorem is, again by virtue of Theorem 17, equivalent to (N). Since (*) is more complicated than either (**) or (N), and since (N) seems closer to "the negative numbers are the opposites of the positive numbers" than is (**), we have chosen (N) as our basic principle for negative numbers.

We do not adjoin (N) to our "official" set of basic principles because, for our purposes, theorems about negative numbers are not of much

intrinsic interest. The purpose of the exercises of Part D is to familiarize students with the use of the substitution rule for biconditional sentences [see the COMMENTARY for page 7-22, particularly "explanation (II)"]. However, Exercise 7, like the exercises of Part C, can be used in section 7.03 [specifically, in solving Exercise 1 of Part B on page 7-38].

*

Answers for Part D.

1. By Theorem 17, $a = --a$. Since, by (N), $--a \in N$ if and only if $-a \in P$, it follows that $a \in N$ if and only if $-a \in P$. Consequently, $\forall_x [x \in N \Longleftrightarrow -x \in P]$.

   [This proof uses the substitution rule for equations, but does not use the substitution rule for biconditional sentences. Students may suggest a longer proof which uses both substitution rules. Such a proof may be obtained by interpolating between the first and second sentences of the proof given above the sentence 'So, $a \in N$ if and only if $--a \in N$.' In complete column form this longer proof is:

(1)	$\forall_x\ x = --x$	[Theorem 17]
(2)	$a = --a$	[(1)]
(3)	$a \in N \Longleftrightarrow a \in N$	[logical principle]
(4)	$a \in N \Longleftrightarrow --a \in N$	[(2), (3)]
(5)	$\forall_x [-x \in N \Longleftrightarrow x \in P]$	[(N)]
(6)	$--a \in N \Longleftrightarrow -a \in P$	[(5)]
(7)	$a \in N \Longleftrightarrow -a \in P$	[(4), (6)]
(8)	$\forall_x [x \in N \Longleftrightarrow -x \in P]$	[(1)-(7)]

   Step (3) is justified by the fact that any sentence of the form '$p \Longleftrightarrow p$' is logically valid. Step (4) follows from (2) and (3) by the substitution rule for equations. Step (7) follows from (4) and (6) by the substitution rule for biconditional sentences--(7) is obtained by replacing, in (4), the left-hand component of the biconditional sentence (6) by its right-hand component. (The fact that (4) happens, also, to be a biconditional sentence is incidental.)]

[Answers for the remaining exercises of Part D are given in the COMMENTARY for page 7-27.]

[illegible]

Correction. On page 7-27, line 3b should read:

12. If --- g(1) = , g(3) = , and ---

4. 50; 15	5. 9; $\frac{8}{3}$		6. -7; -11
7. 96	8. 100	9. 54	10. 35
11. 8; 64; 512	12. $-\frac{7}{4}$; $-\frac{1}{2}$; $-\frac{4}{3}$		13. 2 and -2

[Questions concerning the values of functions like that described in Exercise A 11 for arguments other than positive integers are answered in Units 8 and 9.

In discussing Exercise A 12, ask if g is defined for all real numbers. Ask for a sketch of the graphs of the functions in Exercise A 13.]

*

Answers for Part B.

1. (A), (B)	2. (B), (C)	3. (A), (B)	4. (A)
5. (A)	6. (B), (C)	7. (B)	8. (A), (C)

*

Answers for Part C.

1. 72	2. 64	3. -72	4. 2	5. 18	6. 23

*

Answers for Part D.

1. F	2. F	3. F	4. F	5. F	6. F

[Have students modify the statements in Part D to correct the errors.]

rise to two interpretations for '>' for which all the theorems of section 7.03 are true--in brief, the subfield $\mathcal{S}$ of the real numbers can be ordered in either of two ways. This is because there are two real numbers each of which has 2 as its square and neither of which is rational. And, while our basic principles do determine which rational numbers must belong to P [for example, $1 \in P$ and $-1 \notin P$], they give us no way of distinguishing between the two square roots of 2. Consequently, we are free to put either of them [but, by (P_2), not both] into P.

*

Answers for Miscellaneous Exercises.

[All exercises in Parts A, B, C, D, and E are <u>easy</u> except B5, which is of <u>medium</u> difficulty; F1-4 are <u>medium</u>; F5, 6 are <u>hard</u>.]

Answers for Part A.

1. 210; 620; −10   2. 210; 620; −10   3. 210; 0; 1230

[Exercises A1-3 provide an interesting application of factoring. Others like these can be manufactured easily. Here are a few:

$f(x) = x^2 + 5x - 84$; $f(107) =$, $f(88) =$, $f(-93) =$, $f(-112) =$.

$f(x) = 6x^2 + 29x + 9$; $f(33) =$, $f(3) =$, $f(\frac{1}{2}) =$, $f(8) =$.

$f(x) = \dfrac{15x^2 + 19x - 8}{3x^2 - 28x + 9}$; $f(18) =$, $f(19) =$, $f(109) =$.

$f(x, y) = x^2 - y^2$; $f(93, 92) =$, $f(68, 32) =$.

$f(x, y) = xy^2 - x^2y$; $f(-3, 7) =$, $f(9, 109) =$.

Practice with exercises like these will teach students to be on the lookout for factors.]

*

The notation '$f((x, y))$' is customarily simplified to '$f(x, y)$'. This custom is followed in the succeeding exercises, except for the first occurrence of such an expression in Exercise 4.

*

although $\sqrt{2} \in P$, there is no $y[\in \mathcal{S}]$ such that $\sqrt{2} = y^2$. To see this, suppose that r and s are rational numbers such that $\sqrt{2} = (r + s\sqrt{2})^2$. Then, $\sqrt{2} = r^2 + 2s^2 + 2rs\sqrt{2}$, and $(1 - 2rs)\sqrt{2} = r^2 + 2s^2$. Since $\sqrt{2}$ is not rational, it follows that $1 - 2rs = 0$ and, hence, that $r^2 + 2s^2 = 0$. Now, it follows by (*) that if $r \neq 0$ then $r^2 \in P$; and, since $2 \in P$, it follows by (*) and (P_4) that if $s \neq 0$ then $2s^2 \in P$. Hence, by (P_3), unless $r = 0 = s$, $r^2 + 2s^2 \in P$. Consequently, since $0 \notin P$, it follows that if $r^2 + 2s^2 = 0$ then $r = 0 = s$. But, if $r = 0 = s$ then $1 - 2rs = 1 \neq 0$. So, there are no rational numbers r and s such that $\sqrt{2} = (r + s\sqrt{2})^2$--that is, under the present interpretation, there is no number y such that $\sqrt{2} = y^2$. Hence, under the present interpretation, (**) is false. Since, under the same interpretation, our fifteen basic principles are true, (**) is not a consequence of these principles.

Now, the principal interest in the preceding interpretation is that it is one of a pair of interpretations of our fifteen basic principles which differ only in the interpretation given to 'P'. In fact, if we give the same interpretations to '0', '+', '$\cdot$', '$-$', '-', and '$\div$' [so that the first eleven basic principles are satisfied] we can satisfy (P_1) - (P_4) <u>either</u> by interpreting 'P' as in the preceding paragraph <u>or</u> by interpreting 'P' as denoting the set of those numbers, $r + s\sqrt{2}$, in $\mathcal{S}$ such that $r - s\sqrt{2}$ is a positive real number. [To see that, for example, (P_2) is true under this interpretation, suppose that $a \in P$ and $b \in P$. Then, there are rational numbers r_1, s_1, r_2, and s_2 such that $a = r_1 + s_1\sqrt{2}$, $b = r_2 + s_2\sqrt{2}$, and $r_1 - s_1\sqrt{2}$ and $r_2 - s_2\sqrt{2}$ are positive real numbers. Now, since the sum of positive real numbers is a positive real number, and since $(r_1 - s_1\sqrt{2}) + (r_2 - s_2\sqrt{2}) = (r_1 + r_2) - (s_1 + s_2)\sqrt{2}$, it follows that $(r_1 + r_2) - (s_1 + s_2)\sqrt{2}$ is a positive real number. But, with the present interpretation of 'P' this means that $(r_1 + r_2) + (s_1 + s_2)\sqrt{2} \in P$. So, $a + b \in P$.] In view of section 7.03, the two interpretations of 'P' give

real number has a square root. Since $0^2 = 0 \notin P$, this assumption could be written:

$$(**) \qquad \forall_x [x \in P \Longrightarrow \exists_{y \neq 0}\, x = y^2]$$

Statements (*) and (**) can be combined into one:

$$\forall_x [x \in P \Longleftrightarrow \exists_{y \neq 0}\, x = y^2]$$

[In fact, theorem (*) is equivalent to '$\forall_x \forall_{y \neq 0} [x = y^2 \Longrightarrow x \in P]$' and this is equivalent to '$\forall_x [\exists_{y \neq 0}\, x = y^2 \Longrightarrow x \in P]$.']

In a later unit we shall adopt a completeness principle for the real numbers by the use of which (**) can be proved. That (**) is not a consequence of our present fifteen basic principles can be seen by considering the interpretation in which the domain of the variables is the set of rational real numbers. Since 0 and 1 are rational numbers and since the set of rational numbers is closed with respect to addition, multiplication, opposition, subtraction, and division, it follows that, if '0', '1', '+', '·', '−', '-', and '÷' are given their usual meanings [except that the operations are restricted to rational numbers], we have an interpretation for which the first eleven basic principles are true. And, if 'P' denotes the set of positive rational numbers then (P_1) - (P_4) are true also. But, since there is no rational number whose square is 2, (**) is not true in this interpretation. Hence, (**) is not a consequence of our fifteen basic principles.

In view of the fact that, as is done on page 7-30, > can be defined in terms of P, it is interesting to consider a somewhat more complicated interpretation which not only shows that (**) is not a consequence of our fifteen basic principles but also shows that, in some cases, the domain of the variables in an interpretation of our fifteen basic principles can be ordered in either of two equally satisfactory ways by adopting either of two equally satisfactory interpretations of 'P'. To show this, take as the domain of 'x', 'y', and 'z' the set $\mathfrak{J}$ of those real numbers which are rational linear combinations of the real numbers 1 and $\sqrt{2}$--that is, the numbers $r + s\sqrt{2}$, where r and s are rational real numbers. Obviously, real numbers 0 and 1 belong to $\mathfrak{J}$, and $\mathfrak{J}$ is closed with respect to addition, multiplication, opposition, subtraction, and division. Hence, if '0', '1', '+', '·', '−', '-', and '÷' are given their usual meanings [except that the operations are restricted to $\mathfrak{J}$], we have an interpretation for which the first eleven basic principles are true. And, if 'P' denotes the set of those members of $\mathfrak{J}$ which are positive real numbers then (P_1) - (P_4) are true, also. Now, recalling that $\sqrt{2}$ is irrational, it is easy to see that, under this interpretation, (**) is false. For,

Students less well versed may feel the need to start:

Suppose that $a \in N$ and $b \in N$. Since, by Exercise 1, $a \in N$ if and only if $-a \in P$, and $b \in N$ if and only if $-b \in P$, it follows [by the substitution rule for biconditionals] that $-a \in P$ and $-b \in P$. So, by (P_3), $-a + -b \in P$.

The argument will then be completed by appealing, first to Theorem 18, next to Exercise 1, and, finally, by conditionalizing, discharging the initial assumption, and generalizing.

There is, of course, nothing wrong with this second argument. It is merely inefficient in the same way that one is inefficient who, wishing merely to be rid of his overcoat, first removes his shoes, then takes off his overcoat, and, finally, puts his shoes back on.

6. [The solution is like that of Exercise 5, using Theorem 23 in place of Theorem 18. In this case, only two instances of Exercise 1 are needed.]

7. Suppose that $a \neq 0$. It follows, from the Sample, that either $a \in P$ or $a \in N$. By (P_4), if $a \in P$ then $a^2 \in P$. By Exercise 6, if $a \in N$ then $a^2 \in P$. So [in either case], $a^2 \in P$. Hence, if $a \neq 0$ then $a^2 \in P$. Consequently, $\forall_x [x \neq 0 \Longrightarrow x^2 \in P]$.

   [The argument given above makes use of the dilemma [see page TC[7-22]d]:

$$\left. \frac{a \in P \text{ or } a \in N \qquad a \in P \Longrightarrow a^2 \in P \qquad a \in N \Longrightarrow a^2 \in P}{a^2 \in P} \quad \right]$$

[As an eighth exercise, you might want to suggest the proof of:

$$\forall_x \forall_y \, [(x \in P \text{ and } y \in N) \Longrightarrow xy \in N]$$

You might also ask students whether the converse of this theorem is also a theorem. [It isn't--if $ab \in N$ it may be the case that $a \in N$ and $b \in P$.]

*

☆Remark on the theorem of Exercise 7.--The theorem in question can be restated as:

(*) $$\forall_{y \neq 0} \, y^2 \in P$$

When discussing square roots in Unit 3, we assumed that each positive

should read:

12. If --- $g(1) =$, $g(3) =$, and $g(\frac{1}{2}) =$.

2. Since, by (P_2), not both $a \in P$ and $-a \in P$, and since, by Exercise 1, $a \in N$ if and only if $-a \in P$, it follows [by the substitution rule for biconditional sentences] that not both $a \in P$ and $a \in N$. Consequently, $\forall_x$ not both $x \in P$ and $x \in N$.

3. Since, by the Sample, if $a \neq 0$ then either $a \in P$ or $a \in N$, and since, by (N), $-a \in N$ if and only if $a \in P$, it follows [by the substitution rule for biconditional sentences] that if $a \neq 0$ then either $-a \in N$ or $a \in N$. Consequently, $\forall_x [x \neq 0 \Longrightarrow$ either $x \in N$ or $-x \in N]$.

   [As given above, the proof is not quite complete (but, unless questioned by students, is acceptable). To complete it one should insert, just before 'Consequently', the following:

   Since either $-a \in N$ or $a \in N$ if and only if either $a \in N$ or $-a \in N$, it follows that if $a \neq 0$ then either $a \in N$ or $-a \in N$.

   The so-expanded argument appeals to the fact that each sentence of the form '(p or q) $\Longleftrightarrow$ (q or p)' is logically valid, and to the substitution principle for biconditional sentences.]

4. [This theorem can be derived from Exercise 2 in just the same way as Exercise 3 was, above, derived from the Sample. As in the case of Exercise 3, a complete proof on these lines would appeal to a logical principle--in this case to the fact that each sentence of the form '(p and q) $\Longleftrightarrow$ (q and p)' is logically valid.

   An abbreviated solution derives the theorem of Exercise 4 directly from (P_2) by "biconditional substitution" from instances of (N) and the theorem of Exercise 1. Again, completeness would require a third substitution--substitution from the logically valid sentence '$(-a \in N$ and $a \in N) \Longleftrightarrow (a \in N$ and $-a \in N)$'.]

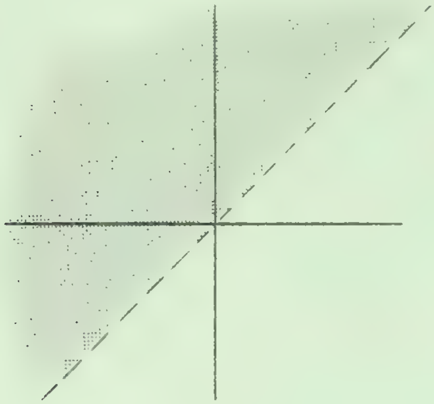
5. Students who are fully aware of the role of the substitution rules should present a proof like this:

   By (P_3), if $-a \in P$ and $-b \in P$ then $-a + -b \in P$. By Theorem 18, $-a + -b = -(a + b)$. So [by the substitution rule for equations], if $-a \in P$ and $-b \in P$ then $-(a + b) \in P$. By Exercise 1, $-a \in P$ if and only if $a \in N$, $-b \in P$ if and only if $b \in N$, and $-(a + b) \in P$ if and only if $a + b \in N$. So [by the substitution rule for biconditional sentences], if $a \in N$ and $b \in N$ then $a + b \in N$. Consequently, $\forall_x \forall_y [(x \in N$ and $y \in N) \Longrightarrow x + y \in N]$.

*

Answers for Exploration Exercises.

1.



2. > [greater than]

Correction. On page 7-29, the section name in the upper left hand corner should be '[7.02]'.

Answers for Part E.

1. $\frac{7}{2}$ 2. 2   3. $-\frac{1}{12}$ 4. 2

*

Answers for Part F.

1, 2. Each line through the center of a regular polygon which has an even number of sides cuts the region bounded by the polygon into two regions with the same area-measure. Students are not expected to prove this--merely to note that the line through the chosen point which passes through the center of the square (Exercise 1) or hexagon (Exercise 2) does the job. However, the proof is not difficult. [Hint. Consider the triangular regions into which the polygonal region is divided by the diagonals through the center of the polygon and a given line through the center. Each of the two polygonal regions into which the line through the center divides the polygonal region is divided, by the diagonals, into triangular regions, and the triangles bounding these regions are congruent in pairs.]

The corresponding problem for an odd-sided polygonal region is much more difficult.

3. $AB \cdot \sqrt{\frac{3\sqrt{3}}{4\pi}}$

4. The two segments are hypotenuses of congruent right triangles. [See Exercise 4 on page 6-156 of Unit 6.] [This problem is nicely generalized by allowing the segments of ℓ and m to be cut off by the lines containing the sides of the square. See problem Q271 on page 120 of the Nov.-Dec. 1960 issue of Mathematics Magazine.]

5. The first player puts his first penny at the center of the table. From then on, he places each of his pennies so that the segment joining its center to the center of the last penny put down by the second player is bisected by the center of the table. [Obviously, the strategy works equally well if the game is played on a square table, or on any table with "central symmetry".]

6. No. Opposite corner squares are both of the same color. On removing them there will be fewer squares of that color than of the other. But each card covers exactly one square of each color.

The material in section 7.03 is hinted at on page 3-106 of Unit 3, and much of it is developed in the accompanying COMMENTARY. However, no acquaintance with this part of Unit 3 is assumed here.

*

As pointed out in the COMMENTARY for page 7-26, although (G) may properly be called a definition of '>', all such definitions must, strictly speaking, be treated as basic principles. [More correctly, must either be treated as basic principles or, like (*) on TC[7-26]e, be supplanted by new basic principles (in the case of (*), the basic principle (N)) from which they can be derived.]

As in Unit 6, we accept some definitions on an informal basis, without introducing the appropriate basic principles. For example, '<' will be used to denote the converse of the relation >, without bothering to state the "enabling principle":

$$\forall_x \forall_y \, [x < y \Longleftrightarrow y > x]$$

as an "official" basic principle. Similarly, we take for granted the definitions:

$$\forall_x \forall_y \, [x \geq y \Longleftrightarrow (x > y \text{ or } x = y)]$$

and:

$$\forall_x \forall_y \, [x \leq y \Longleftrightarrow (x < y \text{ or } x = y)]$$

of '$\geq$' and '$\leq$' in terms of '>'.

There is, of course, no logical necessity for choosing as our primary subject the relation >, rather than <, $\geq$, or $\leq$. Each could easily be defined in terms of the others--for example, if we were to concentrate our attention on $\leq$, '>' would be defined by:

$$\forall_x \forall_y \, [x > y \Longleftrightarrow (y \leq x \text{ and } y \neq x)]$$

Although it is largely unimportant which of the four relations is chosen as "primitive", it is convenient to choose one of them as such, and > is the one which has, in fact, been chosen.

*

As in the statements (P_1) and (P_2), the words 'either' and 'both' in parts <u>a</u> and <u>b</u> of Theorem 86 are pillow-words [see COMMENTARY for page 7-22] and contribute nothing to the meaning of either statement.

*

In some treatments of inequality theorems you may encounter another technique which is sometimes useful. It is based on using the following in place of (G):

$$(G') \qquad \forall_x \forall_y \, [y > x \Longleftrightarrow \exists_{z \in P} \, y = x + z]$$

So, for example, here is how one might prove Theorem 86c:

> Suppose that $a > b$ and $b > c$. Then, by (G′), there are positive numbers m and n such that $a = b + m$ and $b = c + n$. Hence, $a = (c + n) + m = c + (n + m)$. By ($P_3$), $n + m$ is a positive number. So, by (G′), $a > c$. Therefore, if $a > b$ and $b > c$ then $a > c$.

For comparison, here is the "same" proof using (G) instead of (G′):

> Suppose that $a > b$ and $b > c$. Then, by (G), $a - b \in P$ and $b - c \in P$. So, by (P_3), $(a - b) + (b - c) \in P$. Since $a - c = (a - b) + (b - c)$, it follows that $a - c \in P$. So, by (G), $a > c$. Therefore, if $a > b$ and $b > c$ then $a > c$.

Answer for Part B.

(1)	$\forall_x \forall_y\ [y > x \Longleftrightarrow y - x > 0]$	[Theorem 84]
(2)	$0 > a \Longleftrightarrow 0 - a > 0$	[(1)]
(3)	$\forall_x\quad 0 - x = -x$	[Theorem 42]
(4)	$0 - a = -a$	[(3)]
(5)	$0 > a \Longleftrightarrow -a > 0$	[(2), (4)]
(6)	$\forall_x\ [x < 0 \Longleftrightarrow -x > 0]$	[(1) - (5)]

Of course, (5) results from (2) and (4) by an application of the substitution rule for equations. Note the change from '$0 > a$' in (5) to '$x < 0$' in (6). This is an example of the informal treatment of '$<$' referred to in the COMMENTARY for page 7-30. The change from '$>$' to '$<$' might just as well have been made in (2), or in (5), rather than in (6). [As the bracketed sentence in Part B may suggest, Exercise 1 of Part D and Theorem 85 have as one consequence '$\forall_x\ [x < 0 \Longleftrightarrow x \in N]$'.]

*

Answers for Part C.

1. The theorems asked for are boxed at the bottom of page 7-32. The property of $>$ expressed in Theorem 86a is called connectedness; in Theorem 86b, asymmetry; in Theorem 86c, transitivity; in Theorems 86d and 86e, monotoneity, with respect to addition and with respect to multiplication by positive numbers. So, $>$ is connected, asymmetric, transitive, and monotone with respect to addition and to multiplication by positive numbers.

2.

(1)	$\forall_x \forall_y$ not both $x - y \in P$ and $y - x \in P$	[Exercise 3]
(2)	not both $a - b \in P$ and $b - a \in P$	[(1)]
(3)	$\forall_x \forall_y\ [y > x \Longleftrightarrow y - x \in P]$	[(G)]
(4)	$a > b \Longleftrightarrow a - b \in P$	[(3)]
(5)	$b > a \Longleftrightarrow b - a \in P$	[(3)]
(6)	not both $a > b$ and $b > a$	[(2), (4), (5)]
(7)	$\forall_x \forall_y$ not both $x > y$ and $y > x$	[(1) - (6)]

Answers for Part A.

1, 2. The statements asked for are:

$$\forall_x\ [x \neq 0 \Longrightarrow (x > 0 \text{ or } -x > 0)],$$

$$\forall_x\ \text{not both } x > 0 \text{ and } -x > 0,$$

$$\forall_x \forall_y\ [(x > 0 \text{ and } y > 0) \Longrightarrow x + y > 0],$$

$$\forall_x \forall_y\ [(x > 0 \text{ and } y > 0) \Longrightarrow xy > 0],$$

and Theorem 84 [given at the bottom of page 7-31]. It will be sufficient, in most cases, for students merely to write the statements given above in answer to Exercise 1, and to answer 'yes' for Exercise 2. However, it may be worthwhile to go over in class the actual derivation of one of these statements from the corresponding basic principle, and the reverse process of deriving one of the basic principles from the corresponding statement. Whether this is worth doing will depend on whether students need more practice like that in the exercises of Part D on pages 7-26 and 7-27. As a sample, here is a column-derivation of the first statement, above, from (P_1):

(1)	$\forall_x\ [x \neq 0 \Longrightarrow (x \in P \text{ or } -x \in P)]$	[(P_1)]
(2)	$a \neq 0 \Longrightarrow (a \in P \text{ or } -a \in P)$	[(1)]
(3)	$\forall_x\ [x > 0 \Longleftrightarrow x \in P]$	[Theorem 83]
(4)	$a > 0 \Longleftrightarrow a \in P$	[(3)]
(5)	$-a > 0 \Longleftrightarrow -a \in P$	[(3)]
(6)	$a \neq 0 \Longleftrightarrow (a > 0 \text{ or } -a > 0)$	[(2), (4), (5)]
(7)	$\forall_x\ [x \neq 0 \Longrightarrow (x > 0 \text{ or } -x > 0)]$	[(1) - (6)]

Note that step (6) comes from (2), (4), and (5) by two applications of the substitution rule for biconditional sentences.

The reverse derivation is obtained by merely rewriting the steps in the order (7), (6), (3), (4), (5), (2), (1), and changing the marginal comments.

*

Students should thoroughly understand the paragraph which precedes the exercises on page 7-31, and they should feel free to restate a theorem like Theorem 81 in any of its equivalent forms, without comment. The equivalence of '$0 \notin P$' and '$\forall_x [x = 0 \Longrightarrow x \notin P]$' can be shown as indicated in the COMMENTARY for page 7-19.

The restricted quantifiers in the two sentences:

(1) $\forall_{x \in P}\ x \neq 0$ and: $\forall_{x > 0}\ x \neq 0$

are merely convenient devices to abbreviate the quantified conditional sentences:

(2) $\forall_x\ [x \in P \Longrightarrow x \neq 0]$ and: $\forall_x\ [x > 0 \Longrightarrow x \neq 0]$

Statement (1) may be read as 'For each positive number x, x is not 0.' Statement (2) may be read as 'For each number x, if x is positive then x is not 0.'

Students should have no trouble in accepting that statements like (1) and (2) "mean the same thing"; and, if such <u>abbreviatory use</u> were the only use made of restricted quantifiers, little more would need to be said on this subject. Unfortunately, this is not the case--we also use restricted quantifiers as in, for example, the principle of quotients:

(3) $\forall_x \forall_{y \neq 0}\ (x \div y) \cdot y = x$

as a means of restricting the class of instances of a generalization to meaningful sentences. In the language which we have adopted, (3) does not "mean the same thing" as:

(4) $\forall_x \forall_y\ [y \neq 0 \Longrightarrow (x \div y) \cdot y = x]$

In fact, (4) is, for us, meaningless because it has meaningless instances, such as:

$$2^2 - 5 \cdot 2 + 6 \neq 0 \Longrightarrow \frac{1}{2^2 - 5 \cdot 2 + 6}(2^2 - 5 \cdot 2 + 6) = 1$$

[The latter is meaningless because, since $2^2 - 5 \cdot 2 + 6 = 0$, '$\frac{1}{2^2 - 5 \cdot 2 + 6}$' does not name any real number.] The <u>nonabbreviatory use</u> of restricted quantifiers, illustrated in (3), gives rise to some complications in proofs. These are discussed in the COMMENTARY for pages 7-36 through 7-41.

*

So, by virtue of the logical properties of =, Theorem 87 and '$\forall_x \forall_y [x = y \Longrightarrow x \not> y]$' are equivalent.

Of course, one would not expect so detailed an explanation from students. Students, in view of their earlier experiences with Theorem 80 [see last two lines on page 7-18 and first five lines on page 7-19], with Theorem 81 [see page 7-31], and with Exercise 1 of Part A on page 7-24, may say something to the effect that '$\forall_x \forall_y [x = y \Longrightarrow x \not> y]$' <u>follows from</u> Theorem 87 by the substitution rule for equations, and implies Theorem 87 by the principle of identity. [Some may say "Aw, we did that lots of times.". Any who do probably have the right idea.]

*

Students will probably accept, on intuitive grounds, the rule of reasoning, used in proving (*) on page 7-33, according to which, inferences of the form:

$$\frac{\text{not } [p \text{ and } q]}{q \Longrightarrow \text{not } p}$$

are valid. This rule of reasoning is justified in Part A on pages 7-34 and 7-35. In the accompanying COMMENTARY it is shown that, as one would also expect, inferences of the form:

$$\frac{q \Longrightarrow \text{not } p}{\text{not } [p \text{ and } q]}$$

are also valid.

*

Theorem 88a does not appear among the theorems given at the end of this unit. But, it is the only-if-part of Theorem 88 on page 7-150. The if-part of Theorem 88 is the subject of Exercise 1 of Part B on page 7-35.

*

The explanation asked for on page 7-33 in connection with Theorem 88a is that, by the rule of the dilemma [see COMMENTARY for page 7-22], the inference:

$$\frac{b > a \text{ or } b = a \qquad b > a \Longrightarrow a \not> b \qquad b = a \Longrightarrow a \not> b}{a \not> b}$$

is valid. Since we have agreed [see COMMENTARY for page 7-30] to abbreviate '$b > a$ or $b = a$' to '$b \geq a$', it follows, by conditionalizing and discharging, that '$b \geq a \Longrightarrow a \not> b$' is a consequence of the other two premisses of the dilemma. Since these are consequences of Theorem 86b and of Theorem 87, respectively, Theorem 88a <u>is</u> a theorem.

The theorem referred to in the bracket which begins two lines below the first box is Theorem 81 [$0 \notin P$]. By (G), $a > a$ if and only if $a - a \in P$-- that is, if and only if $0 \in P$. Since, by Theorem 81, this is not the case, it follows that $a \ngtr a$. Consequently, $\forall_x\, x \ngtr x$.

The derivation of Theorem 87 from Theorem 86 goes as follows:

By Theorem 86b, not both $a > a$ and $a > a$. Hence, $a \ngtr a$. Consequently, $\forall_x\, x \ngtr x$.

*

The explanation asked for in the line just before (*) on page 7-33 should, by now, be familiar to students. The derivation of:

$$\forall_x \forall_y\, [x = y \Longrightarrow x \ngtr y]$$

from:

$$\forall_x\, x \ngtr x$$

is just like the derivation of (*) of Exercise 1 in Part A on page 7-24 from '$\forall_x\, x - x = 0$'. The reverse derivation is, as noted in the COMMENTARY for page 7-24, even simpler. [See, also, the COMMENTARY for page 7-19.]

In general, if F(a, b) is any sentence [say '$a \ngtr b$'] then the sentence F(a, a) ['$a \ngtr a$'] is equivalent to the sentence $\forall_y\, [a = y \Longrightarrow F(a, y)]$ ['$\forall_y\, [a = y \Longrightarrow a \ngtr y]$']. That each of the two sentences is derivable from the other is shown by the tree-form derivations given below.

$$\dfrac{\dfrac{\dfrac{\overset{*}{a = b} \qquad F(a, a)}{F(a, b)}}{a = b \Longrightarrow F(a, b)}\,{}^{*}}{\forall_y\, [a = y \Longrightarrow F(a, y)]} \qquad\qquad \dfrac{\dfrac{\forall_x\, x = x}{a = a} \qquad \dfrac{\forall_y\, [a = y \Longrightarrow F(a, y)]}{a = a \Longrightarrow F(a, a)}}{F(a, a)}$$

[For explanations of diagrams like these, see Appendix of Unit 6.]

In particular, returning to the explanation asked for in the text, one might say:

Suppose that $a = b$. Since, by Theorem 87, $a \ngtr a$, it follows [using the substitution rule for equations] that $a \ngtr b$. Hence, if $a = b$ then $a \ngtr b$. Consequently, $\forall_x \forall_y\, [x = y \Longrightarrow x \ngtr y]$.

Conversely, it follows from the theorem just proved that if $a = a$ then $a \ngtr a$. Since, by the principle of identity, $a = a$, it follows that $a \ngtr a$. Consequently, $\forall_x\, x \ngtr x$.

The second of the two proofs of (**) which are given on page 7-34, and the proof of (***) which is asked for on the same page, are similar to some proofs in geometry in which a theorem is used in proving its converse [see, for example, the proof of Theorem 4-7 on page 6-117 of Unit 6]. The nature of proofs of this kind is laid bare in Part H on page 7-41. [Taking the function f of Part H to be the "adding c function" the theorem of Exercise 1 of Part H becomes (**). Taking f to be the "multiplying by c function" [for c > 0], the same theorem yields (***). For details, see COMMENTARY for Part H on page 7-41.]

*

The second proof of (**) involves two applications of a principle of logic called <u>the complex dilemma</u>:

$$\frac{p \text{ or } q \qquad p \Rightarrow r \qquad q \Rightarrow s}{r \text{ or } s}$$

This is easily justified on the basis of the basic rules for alternation sentences [page 6-393]. Here's how:

$$\dfrac{p \text{ or } q \qquad \dfrac{\dfrac{\dfrac{\overset{*}{p} \qquad p \Rightarrow r}{r}}{r \text{ or } s}}{p \Rightarrow [r \text{ or } s]}{}^{*} \qquad \dfrac{\dfrac{\dfrac{\overset{*}{q} \qquad q \Rightarrow s}{s}}{r \text{ or } s}}{q \Rightarrow [r \text{ or } s]}{}^{*}}{r \text{ or } s}$$

This rule of reasoning is used in the argument given in the third, fourth, and fifth sentences of the proof:

$$\frac{b \geq a \qquad b = a \Rightarrow b + c = a + c \qquad b > a \Rightarrow b + c > a + c}{b + c \geq a + c}$$

A special case of the rule is also used in the second sentence:

$$\frac{a = b \text{ or } a \neq b \qquad a \neq b \Rightarrow (a > b \text{ or } b > a)}{a = b \text{ or } a > b \text{ or } b > a}$$

Here, the middle premiss, which would be '$a = b \Rightarrow a = b$', is not stated. Such implicit use is permissible since each sentence of the form '$p \Rightarrow p$' is logically valid [see COMMENTARY for page 6-395]. For that matter, the premiss '$a = b$ or $a \neq b$' might have been omitted on the same grounds [see 6-394, and corresponding COMMENTARY].

Incidentally, an argument of the same form:

$$\frac{p \text{ or not } p \qquad p \Longrightarrow p \qquad \text{not } p \Longrightarrow q}{p \text{ or } q}$$

justifies inferences of the form:

$$\frac{\text{not } p \Longrightarrow q}{p \text{ or } q}$$

The reverse inferences--those of the form:

$$\frac{p \text{ or } q}{\text{not } p \Longrightarrow q}$$

can be justified by using the rule for denying an alternative [see page 6-394].

*

Here is a proof of (***):

Suppose that $c > 0$ and that $ac > bc$. Since $a = b$ or $a \neq b$, it follows from Theorem 86a that either $a = b$ or $a > b$ or $b > a$--that is, that either $a > b$ or $b \geq a$. Suppose that $b \geq a$. If $b = a$ then $bc = ac$, and if $b > a$ then, since $c > 0$, it follows from Theorem 86e that $bc > ac$. So, if $b \geq a$ then $bc \geq ac$. But, by Theorem 88a, if $bc \geq ac$ then $ac \not> bc$. Hence, if $b \geq a$ then $ac \not> bc$. Since, by hypothesis, $ac > bc$, it follows that $b \not\geq a$. But, as shown above, either $a > b$ or $b \geq a$. So, $a > b$. Therefore, if $c > 0$ and $ac > bc$ then $a > b$. Consequently, $\forall_x \forall_y \forall_z [(z > 0 \text{ and } xz > yz) \Longrightarrow x > y]$.

*

For answers for Part A, see COMMENTARY for page 7-35.

$(>_1)$ $\forall_x\ [x \neq 0 \Longrightarrow (x > 0 \text{ or } -x > 0)]$

$(>_2)$ $\forall_x$ not both $x > 0$ and $-x > 0$

$(>_3)$ $\forall_x \forall_y\ [(x > 0 \text{ and } y > 0) \Longrightarrow x + y > 0]$

$(>_4)$ $\forall_x \forall_y\ [(x > 0 \text{ and } y > 0) \Longrightarrow xy > 0]$

$(>_5)$ $\forall_x \forall_y\ [y > x \Longleftrightarrow y - x > 0]$

To do so, it is convenient to begin by proving $(>_5)$:

By Theorem 86d, if $b > a$ then $b + -a > a + -a$--that is, $b - a > 0$. Conversely, if $b - a > 0$ then, by Theorem 86d, $b - a + a > 0 + a$--that is, $b > a$. Consequently, $(>_5)$.

Having proved $(>_5)$ we may use it in proving the others. More particularly, we may use its consequence:

$$\forall_x\ [0 > x \Longleftrightarrow -x > 0]$$

Using this, it is easy to derive $(>_1)$ from Theorem 86a, and $(>_2)$ from Theorem 86b.

Statement $(>_3)$ follows from Theorem 86d and 86c in the same way as did the theorem of Exercise 2 of Part C. $(>_4)$ follows easily from Theorem 86e.

*

Part ☆D shows that, in place of (P_1) - (P_4) and (G), we might have adopted Theorem 83 and the five parts of Theorem 86 as basic principles. For, as shown in Part ☆D and Part A on page 7-31, (P_1) - (P_4) and (G) can be derived from our first eleven basic principles, Theorem 86, and Theorem 83. Moreover, 'P' does not occur in Theorem 86, and Theorem 83 can be interpreted as purely a contextual definition of 'P' in terms of '>'. Consequently, if we were interested only in >, to the exclusion of P, it would be sufficient to take, as basic principles, our first eleven together with the five parts of Theorem 86. From these we could derive all those theorems not containing 'P' which can be derived from our present sixteen basic principles.

*

Answers for Part C.

1. By Theorem 84, $a+1 > a$ if and only if $a+1-a > 0$. But, $a+1-a=1$, and, by Theorems 82 and 83, $1 > 0$. So, $a + 1 > a$. Consequently, $\forall_x\, x + 1 > x$.

   [Another proof starts with '$1 > 0$' and makes use of Theorem 86d (or of Theorem 89).]

2. Suppose that $a > b$ and that $c > d$. It follows from Theorem 86d that $a + c > b + c$ and that $b + c > b + d$. So, by Theorem 86c, $a + c > b + d$. Hence, if $a > b$ and $c > d$ then $a + c > b + d$. Consequently, $\forall_x \ldots$.

3. Suppose that $a > b$ and that $b \geq c$. Since $b \geq c$, either $b = c$ or $b > c$. If $b = c$ then, since $a > b$, $a > c$. If $b > c$ then, since $a > b$, it follows from Theorem 86c that $a > c$. So, in either case, $a > c$. Hence, if $a > b$ and $b \geq c$ then $a > c$. Consequently, $\forall_x \ldots$.

[One can go a step further, and use this theorem to prove '$\forall_x \forall_y \forall_z\, [(x \geq y$ and $y \geq z) \Longrightarrow x \geq z]$'. One can also prove similar modifications of, for example, Theorem 91.]

4. Suppose that $a \neq b$. By Theorem 86a, it follows that either $a > b$ or $b > a$. Suppose that $a > b$. It then follows, by Theorem 88a, that $b \not\geq a$. Since it is not the case that $b \geq a$, it is not the case that both $a \geq b$ and $b \geq a$. Similarly, assuming that $b > a$, it follows that it is not the case that both $a \geq b$ and $b \geq a$. So, in any case, if $a \neq b$ then not both $a \geq b$ and $b \geq a$. Hence, if $a \geq b$ and $b \geq a$ then $a = b$. Consequently, $\forall_x \ldots$.

5. By Theorem 89, $-a > -b$ if and only if $-a + (a + b) > -b + (a + b)$--that is, if and only if $b > a$. Consequently, $\forall_x \ldots$.

   [There are several minor variations of this proof.]

*

Answer for Part ☆D.

The problem is to derive, from Theorem 86 [and theorems derivable from the first eleven basic principles], the following five statements:

Answers for Part A.

1. '$a > b$' [p], and '$b > a$' [q].

2. The first inference is one of the basic inferences for conjunction sentences, the second is conditionalizing, with attendant discharge of a premiss, the third is modus tollens, and the last is conditionalizing, again accompanied by discharge of a premiss.

*

Students may wonder how to justify inferences of the form:

$$\frac{p \Rightarrow \text{not } q}{\text{not } [p \text{ and } q]}$$

This can be done by appealing to a type of argument, akin to proof by contradiction [see Unit 6 COMMENTARY for page 6-386], called <u>refutation by contradiction</u>. Here's how:

$$\frac{\dfrac{\overset{*}{p \text{ and } q}}{q} \qquad \dfrac{\dfrac{\overset{*}{p \text{ and } q}}{p} \qquad p \Rightarrow \text{not } q}{\text{not } q}}{\text{not } [p \text{ and } q]}\,{}^{*}$$

One can also turn this into an actual proof by contradiction by using the reverse rule of double denial to derive each of the premisses 'p and q' in the scheme given above from 'not (not [p and q])'.

*

Answers for Part B.

1. The theorem to be proved is '$\forall_x \forall_y\, [x \not> y \Rightarrow y \geq x]$'.

   Since $a = b$ or $a \neq b$, it follows from Theorem 86a that either $a = b$ or $a > b$ or $b > a$--that is, that either $b \geq a$ or $a > b$. So, if $a \not> b$ then $b \geq a$. Consequently, $\forall_x \ldots$.

2. Suppose that $a + c > b + c$. It follows, by Theorem 86d, that $a + c + -c > b + c + -c$. So, since $a + c + -c = a + (c + -c) = a + 0 = a$ and, similarly, $b + c + -c = b$, it follows that $a > b$. Hence, if $a + c > b + c$ then $a > b$. Consequently, $\forall_x \ldots$.

To call to mind the inequation transformation principles and to illustrate their usefulness, precede the discussion on this page by having the class go through the solutions of the following inequations:

(a) $4x - 5 > 3x + 7$ (b) $6x - 5 > 4x + 1$

(c) $11x + 2 > 14x + 8$ (d) $x^2 + 6x - 7 > 0$

(e) $x^2 - 7x + 10 > 0$

In discussing the equivalence of successive inequations in the solutions, you should be able to bring in many of the comments which follow as well as the importance of the biconditional nature of the transformation principles.

*

There follow some comments on solving inequations and on proving generalizations for inequations.

The transformation principles for inequations can be used in transforming an inequation which one wishes to solve into an equivalent one which is easier to solve. [See Unit 3 for the corresponding process of solving equations.] For example, to solve '$3x + 2 > 7 - x$', one notes that, by the atpi, this is equivalent to [that is, has the same solutions as]:

$$3x + 2 + x > 7 - x + x$$

Processes of algebraic simplification show that this is equivalent to:

$$4x + 2 > 7$$

By the atpi and algebraic simplification, the last is equivalent to:

$$4x > 5$$

and, by the pq and the cpm, this is equivalent [since $4 \neq 0$] to:

$$4x > 4 \cdot \frac{5}{4}$$

Finally [since $4 > 0$], this is, by the mtpi, equivalent to:

$$x > \frac{5}{4}$$

Hence, the solution set of '$3x + 2 > 7 - x$' is $\{x: x > \frac{5}{4}\}$.

The point to be noted is that since each of the succeeding inequations is equivalent to the preceding one, the last, '$x > 5/4$' is equivalent to the given one, '$3x + 2 > 7 - x$'. So, its solution set is the sought-for solution set of the given inequation.

[illegible]

The procedure of solving an equation or inequation by starting with the given sentence and modifying either or both sides until one obtains a sentence which is easier to solve, is often criticized. Such criticism is justified if one or more of the succeeding sentences is merely a consequence of the preceding one--for, in this case, the later sentences may have solutions which fail to satisfy the given sentence. However, if each succeeding sentence is known to be equivalent to the preceding sentence then such criticism is invalid.

As an illustration of this point, consider the following sketch for a proof of '$\forall_x\ x + 1 > x$':

(a)	$x + 1 > x$
(b)	$x + 1 > x + 0$
(c)	$1 > 0$
(d)	$\forall_x\ x + 1 > x$

A person writing this might have in mind that (a) is equivalent to (b) by virtue of the pa0 and the substitution rule for equations and that (b) is equivalent to (c) by virtue of the atpi [and, strictly speaking, the cpa and the substitution principle for equations]. So, since (c) is a theorem [Theorems 82 and 83], it follows that $\forall_x\ x + 1 > x$. In column form:

(1)	$\forall_x\ x + 0 = x$	[pa0]
(2)	$a + 0 = a$	[(1)]
(3)	$a + 1 > a \Longleftrightarrow a + 1 > a$	[logical principle]
(4)	$a + 1 > a \Longleftrightarrow a + 1 > a + 0$	[(2), (3)]
(5)	$\forall_x \forall_y \forall_z\ [z + x > z + y \Longleftrightarrow x > y]$	[theorem]
(6)	$a + 1 > a + 0 \Longleftrightarrow 1 > 0$	[(5)]
(7)	$a + 1 > a \Longleftrightarrow 1 > 0$	[(4), (6)]
(8)	$1 > 0$	[theorem]
(9)	$a + 1 > a$	[(7), (8)]
(10)	$\forall_x\ x + 1 > x$	[(1) - (9)]

With this interpretation, (a)-(d) outlines a completely valid proof of '$\forall_x\, x + 1 > x$'. However, in the absence of explanatory remarks, (a)-(d) may be interpreted as an outline of an invalid argument whose first four steps are (1)-(4) as above but which then proceeds as follows:

(5′)	$\forall_x \forall_y \forall_z\, [z + x > z + y \Longrightarrow x > y]$	[theorem]
(6′)	$a + 1 > a + 0 \Longrightarrow 1 > 0$	[(5′)]
(7′)	$a + 1 > a \Longrightarrow 1 > 0$	[(4), (6′)]
(8′)	$1 > 0$	[theorem]
(9′)	$a + 1 > a$	[(7′), (8′)]
(10′)	$\forall_x\, x + 1 > x$	[(1) - (9′)]

The inference of (9′) from (7′) and (8′) is, of course, invalid. It is an example of the fallacy of affirming the consequent [see page 6-378 of Unit 6]. The corresponding inference of (9) from (7) and (8) is, on the other hand, valid. It is an example of the use of the substitution rule for biconditional sentences.

Students should be allowed to make use of some short-hand like (a) - (d) for outlining proofs which proceed by transforming sentences into equivalent sentences and ending up with a previously proved theorem. One which seems safe to use is illustrated by:

$$\left.\begin{aligned} & x + 1 > x \\ \Longleftrightarrow\ & x + 1 > x + 0 \\ \Longleftrightarrow\ & 1 > 0 \end{aligned}\right\}\ \begin{aligned} &\text{pa0 [or: algebra]} \\ &\text{atpi} \end{aligned}$$

But, $1 > 0$. So, $\forall_x\, x + 1 > x$.

[Similar remarks apply to proving a trigonometric identity by transforming both sides.]

*

The proof of the multiplication transformation principle furnishes a convenient point at which to begin illustrating the ways in which restricted quantifiers occur in proofs. To begin with, let's see how one might formalize a derivation of the only if-part of part a of the mtpi from (***) on page 7-34.

(1) $\forall_x \forall_y \forall_z [(z > 0 \text{ and } xz > yz) \Longrightarrow x > y]$ [(***)]

(2) $(c > 0 \text{ and } ac > bc) \Longrightarrow a > b$ [(1)]

(3) $ac > bc$ [assumption]*

(4) $c > 0 \text{ and } ac > bc$ [c > 0] [(3)]

(5) $a > b$ [(2), (4)]

(6) $ac > bc \Longrightarrow a > b$ [(5); *(3)]

(7) $\forall_x \forall_y \forall_{z > 0} [xz > yz \Longrightarrow x > y]$ [(1) - (6)]

The new things occurring in this proof are the restriction '[c > 0]' on step (4) and the restricted quantifier '$\forall_{z > 0}$'. Up to this time, we could have accepted (4) as a consequence of two preceding steps--'c > 0' and 'ac > bc'. We are now, in order to formalize the procedure of introducing restricted quantifiers, adopting a new principle of logic:

(R_1) A premiss needed for a step in a proof may be introduced as a restriction on the step itself, rather than as an assumption.

A restriction on a step in a test-pattern [for example, the restriction on step (4) in the test-pattern (1) - (6)] should be thought of as a restriction on the permissible substitutions for the variables. Thus, the restricted test-pattern, above, serves as a pattern for verifying only instances of the restricted generalization (7), and not the additional instances of the unrestricted generalization '$\forall_x \forall_y \forall_z [xz > yz \Longrightarrow x > y]$'. Hence, by the test-pattern principle, (1) - (6) justify (7), but not the unrestricted generalization. So, in applying the test-pattern principle one must now make sure that the quantifiers in the concluding sentence are restricted in such a way as to show the restrictions which have been applied to the steps in the test-pattern.

Note that (R_1) is permissive--we chose to introduce 'c > 0' as a restriction, rather than as an assumption, because we wished to establish (7), rather than, say:

$$\forall_x \forall_y \forall_z [z > 0 \Longrightarrow (xz > yz \Longrightarrow x > y)]$$

Later we shall introduce a prescriptive rule, (R_2), whose adoption is necessitated by our nonabbreviatory use of restricted quantifiers.

[The proof given above justifies the only if-part of part a of the mtpi. To establish the mtpi as a whole, one would omit step (7) and continue with a 6-step derivation of '$a > b \Longrightarrow ac > bc$' from Theorem 86e. The

thirteenth step would, then, be '$ac > bc \Longleftrightarrow a > b$' [a consequence of (6) and (12)] and the fourteenth would be the mtpi, itself, justified by the test-pattern (1)-(13). For this proof the restriction '[$c > 0$]' would occur again, this time at step (10).]

Whether a sentence in a column proof is an assumption [which, typically, is to be discharged on conditionalizing] or a restriction [which is taken note of by restricting a quantifier when generalizing] is made clear by the position which the sentence occupies [and by the absence or presence of brackets]. In a paragraph proof, it is convenient to signalize the distinction by using 'suppose' when introducing an assumption and 'for' when introducing a restriction. Thus, the preceding proof of the only-if-part of part a of the mtpi might be paragraphed as follows:

> From (***) it follows that if $c > 0$ and $ac > bc$ then $a > b$. Suppose that $ac > bc$. Then, for $c > 0$, $a > b$. Hence, [for $c > 0$], if $ac > bc$ then $a > b$. Consequently, $\forall_x \forall_y \forall_{z>0} [xz > yz \Longrightarrow x > y]$.

The preceding example has illustrated how, by introducing restrictions in a test-pattern, one can obtain restricted generalizations as conclusions. Let's, now, see how restricted generalizations are used as premisses. As an example it will be convenient to retrace our steps and derive (***) of page 7-34 from the only-if-part of part a of the mtpi.

(1)	$\forall_x \forall_y \forall_{z>0} [xz > yz \Longrightarrow x > y]$	[mpti]
(2)	$c > 0$ and $ac > bc$	[assumption]*
(3)	$c > 0$	[(2)]
(4)	$ac > bc \Longrightarrow a > b$	[(1), (3)]
(5)	$ac > bc$	[(2)]
(6)	$a > b$	[(4), (5)]
(7)	$(c > 0$ and $ac > bc) \Longrightarrow a > b$	[(6); *(2)]
(8)	$\forall_x \forall_y \forall_z [(z > 0$ and $xz > yz) \Longrightarrow x > y]$	[(1)-(7)]

The new thing here is the need for '$c > 0$' [step (3)], as well as (1), as a premiss from which to conclude (4). The need is clearly due to the presence of the restricted quantifier '$\forall_{z>0}$'--rather than '$\forall_z$'--in (1). The principle of logic involved may be called the rule for restricted universal instantiation:

Each instance of a restricted universal generalization sentence is a consequence of the generalization together with an auxiliary premiss which states the restriction.

In the example above, the auxiliary premiss '$c > 0$' for step (4) is a consequence of step (2) and, so, is necessarily a step in the proof. In other cases, an auxiliary hypothesis may be subject to the rule (R_1) and, so, may appear either as an assumption--hence, as a step in the proof--or as a restriction, depending on what conclusion one is aiming at.

As a concluding example, here is a translation of the paragraph proof given on pages 7-36 and 7-37 of part <u>b</u> of the mtpi.

(1)	$\forall_x\, [x < 0 \Longleftrightarrow -x > 0]$		[Theorem 85]
(2)	$c < 0 \Longrightarrow -c > 0$		[(1)]
(3)	$-c > 0$	[$c < 0$]	[(2)]
(4)	$\forall_x \forall_y \forall_{z > 0}\, [xz > yz \Longleftrightarrow x > y]$		[mtpi part <u>a</u>]
(5)	$a \cdot -c > b\ -c \Longleftrightarrow a > b$		[(3), (4)]
(6)	$\forall_x \forall_y\ -(xy) = x \cdot -y$		[Theorem 20]
(7)	$-(ac) = a \cdot -c$		[(6)]
(8)	$-(bc) = b \cdot -c$		[(6)]
(9)	$-(ac) > -(bc) \Longleftrightarrow a > b$		[(5), (7), (8)]
(10)	$\forall_x \forall_y\, [-x > -y \Longleftrightarrow y > x]$		[Theorem 94]
(11)	$-(ac) > -(bc) \Longleftrightarrow bc > ac$		[(10)]
(12)	$ac < bc \Longleftrightarrow a > b$		[(9), (11)]
(13)	$\forall_x \forall_y \forall_{z < 0}\, [xz < yz \Longleftrightarrow x > y]$		[(1) - (12)]

[In deriving (2) from (1), the step '$c < 0 \Longleftrightarrow -c > 0$' has been omitted. In deriving (12) from (9) and (11), '$bc > ac$' has, as suggested on page 7-30, been replaced by '$ac < bc$', without comment.]

As a simple exercise to illustrate the introduction of restrictions, derive '$\forall_{x \in P}\ x \neq 0$' from '$\forall_x\, [x \in P \Longrightarrow x \neq 0]$' [four steps]. Then, carry out the reverse derivation [five steps].

[Note that, although we cited Theorem 91, a slightly stronger theorem was actually used. The situation is analogous to citing Theorem 86c and actually using Theorem 92. At the same time we also used, without mention, the pa0--specifically, the fact that $0 + 0 = 0$. In the interests of brevity, students should feel free, just as in the proofs in Unit 6, to allow minor gaps, such as those just noted, in their proofs. As in Unit 6, they should be aware of such gaps.]

There is an alternative proof for (1):

> Suppose that $b > a$. Then, for $a \geq 0$, it follows, by Theorem 92, that $b > 0$. So, by Theorem 86e [$z = a$], $ab \geq a^2$ and, by the same theorem [$z = b$], $b^2 > ab$. Hence, by Theorem 92, $b^2 > a^2$. So, for $a \geq 0$, ...

[Here, again, the first citation of Theorem 86e should be to a slightly different theorem: $\forall_x \forall_y \forall_z [(z \geq 0 \text{ and } x > y) \Longrightarrow xz \geq yz]$]

The proofs of (2) and (3) are quite simple and are similar to each other. Here is a proof of (2):

> Suppose that $b > a$. Then, by Theorem 84, $b - a > 0$. Now, similarly, for $b > -a$, $b + a > 0$. Hence, for $b > -a$, $(b - a)(b + a) > 0$. Since ... .

6. The theorems called for have already been stated and proved in the COMMENTARY for Exercises 4 and 5.

*

Note that since, by definition, $\forall_{x \geq 0} (\sqrt{x} > 0 \text{ and } [\sqrt{x}]^2 = x)$, it follows from Theorems 98b and 98c that $\forall_{x \geq 0} \forall_{y \geq 0} [y > x \Longleftrightarrow \sqrt{y} > \sqrt{x}]$.

4. $(-4)^2 > 3^2$, but $-4 \not> 3$. However, the following is a theorem:

$$\forall_x \forall_{y \geq 0} [y^2 > x^2 \Longrightarrow y > x > -y]$$

Suppose that $b^2 > a^2$. It follows, by Theorem 84, that $b^2 - a^2 > 0$. Since $b^2 - a^2 = (b - a)(b + a)$, it follows that $(b - a)(b + a) > 0$. So, by part a of the ftpi [Theorem 96], either $(b - a > 0$ and $b + a > 0)$ or $(b - a < 0$ and $b + a < 0)$. Hence, either $(b > a$ and $a > -b)$ or $(b < a$ and $b < -a)$.

Suppose that $b < a$ and $b < -a$. Then, for $b \geq 0$, it follows, by Theorem 92, that $a > 0$ and $-a > 0$. But, by (P_2), this is impossible.

Hence [for $b \geq 0$], if $b^2 > a^2$, the only alternative is that $(b > a$ and $a > -b)$. Consequently, $\forall_x \forall_{y \geq 0} [y^2 > x^2 \Longrightarrow y > x > -y]$.

*

In the paragraph proofs for the theorems of Exercises 3 and 4 we have referred to (P_2) and have made implicit use of Theorem 83. Instead of doing this, we could have referred to Theorem 86b and Theorem 85 from which it follows that, for each x, not both $x > 0$ and $-x > 0$.

*

5. $3 > -4$, but $3^2 \not> (-4)^2$. However,

(1) $$\forall_{x \geq 0} \forall_y [y > x \Longrightarrow y^2 > x^2].$$

Strictly speaking, more knowledge about b (alone) will not help. However, if $b > -a$ [as well as $> a$] then $b^2 > a^2$--

(2) $$\forall_x \forall_{y > -x} [y > x \Longrightarrow y^2 > x^2].$$

Equivalently, it is also the case that

(3) $$\forall_y \forall_{x > -y} [y > x \Longrightarrow y^2 > x^2].$$

Exercises 3 and 4 should predispose students to discover (1). This they should do. Whether they also discover (2) or (3) is not important.

Here is a proof of (1):

Suppose that $b > a$. Then, by Theorem 84, $b - a > 0$. Also, for $a \geq 0$, it follows from Theorem 92 that $b > 0$. Hence, by Theorem 91, for $a \geq 0$, $b + a > 0$. So, by the ftpi [or (P_4), or Theorem 86e], $(b - a)(b + a) > 0$. Since $(b - a)(b + a) = b^2 - a^2$, $b^2 - a^2 > 0$. Hence, by Theorem 84, $b^2 > a^2$. So, for $a \geq 0$, if $b > a$ then $b^2 > a^2$. Consequently, $\forall_{x \geq 0} \forall_y [y > x \Longrightarrow y^2 > x^2]$.

[The principle of logic used in deducing (20) from (3), (11), and (19) is, of course, the dilemma. The restriction on the quantifier comes from step (3) for sentence (1a).]

Sentence (4b):

(21)	$a^2 > 0$	[(4), (12), (20)]
(22)	$a \neq 0 \Longrightarrow a^2 > 0$	[(21); *(3)]
(23)	$\forall_x [x \neq 0 \Longrightarrow x^2 > 0]$	[(1) - (22)]

[Because (1b) involves one more step than (1a), the reference numbers are uniformly 1 larger here than in the discussion of sentence (4a).]

*

2. The theorem in question is:

$$\forall_x \forall_y [x^2 = y^2 \Longrightarrow (x = y \text{ or } x = -y)]$$

Here is a paragraph proof:

Suppose that $a^2 = b^2$. By (*), on page 7-24, it follows that $a^2 - b^2 = 0$. So, since $a^2 - b^2 = (a - b)(a + b)$, it follows that $(a - b)(a + b) = 0$. Hence, by the 0-product theorem [Theorem 56], either $a - b = 0$ or $a + b = 0$--that is, either $a = b$ or $a = -b$. So, if $a^2 = b^2$ then either $a = b$ or $a = -b$. Consequently, $\forall_x \forall_y [x^2 = y^2 \Longrightarrow (x = y \text{ or } x = -y)]$.

3. $\forall_{x \geq 0} \forall_{y \geq 0} [x^2 = y^2 \Longrightarrow x = y]$

Suppose that $a^2 = b^2$. By the theorem of Exercise 2, either $a = b$ or $a = -b$.

Suppose that $a = -b$. By (P_2), not both b and $-b$ can be positive--that is, b and a cannot both be positive. So, for $a \geq 0$ and $b \geq 0$, either $a = 0$ or $b = 0$. If $a = 0$ then, since $a = -b$, $-b = 0$ and, by Theorem 79, $b = 0$. So, if $a = 0$ then $a = b$. Similarly, if $b = 0$ then [by Theorem 80] $a = 0$. So, if $b = 0$ then $a = b$. Hence, [for $a \geq 0$ and $b \geq 0$], if $a = -b$ then $a = b$.

So [for $a \geq 0$ and $b \geq 0$], if $a^2 = b^2$ then $a = b$. Consequently, $\forall_{x \geq 0} \forall_{y \geq 0} [x^2 = y^2 \Longrightarrow x = y]$.

Another proof of this theorem is noted in the COMMENTARY for Part H on page 7-41.

Sentence (2a):

$$
\begin{array}{lll}
(4) & a > 0 & [\text{assumption}]^* \\
(5) & \forall_x \forall_y \forall_{z>0} [xz > yz \iff x > y] & [\text{Theorem 95a}] \\
(6) & aa > 0a \iff a > 0 & [(4), (5)] \\
(7) & \forall_x\ 0x = 0 & [\text{theorem}] \\
(8) & 0a = 0 & [(7)] \\
(9) & a^2 > 0 \iff a > 0 & [(6), (8)] \\
(10) & a^2 > 0 & [(4), (9)] \\
(11) & a > 0 \Longrightarrow a^2 > 0 & [(10);\ *(4)]
\end{array}
$$

Notice that, because of the restricted quantifier in (5), (4) is needed, as well as (5), if one is to infer (6). It would still be needed to obtain (6) if (5) were replaced by:

$$\forall_x \forall_y \forall_z [z > 0 \Longrightarrow (xz > yz \iff x > y)]$$

[In replacing 'aa' in (6) by 'a^2' in (9), we are using 'a^2' informally as an abbreviation for 'aa'. In Unit 8 more attention is paid to exponents.] [Notice that, although (11) is an immediate consequence of (9), inferring (11) directly from (9) would leave it subject to the as yet undischarged assumption (4). So, instead, we infer (10) from (4) and (9) by biconditional substitution.]

Sentence (3a) can be expanded into a very similar sequence of steps, (12) - (19). [Of these, (18) would be '$0 < a^2$' but, following the advice given on page 7-30, could more conveniently be written as '$a^2 > 0$'.]

Sentences (2b) and (3b) are, as previously mentioned, like (2a) and (3a).

Finally, consider sentences (4a) and (4b).

Sentence (4a):

$$
\begin{array}{lll}
(20) & a^2 > 0 & [(3), (11), (19)] \\
(21) & \forall_{x \neq 0}\ x^2 > 0 & [(1) - (20)]
\end{array}
$$

A similar proof of the (b)-form of the theorem would differ only in the first and last sentences:

(1b) Suppose that $a \neq 0$. Then, by Theorem 86a, either $a > 0$ or $0 > a$.

- - - -

- - - -

(4b) Since, in either case, $a^2 > 0$, it follows that if $a \neq 0$ then $a^2 > 0$. Consequently, $\forall_x \, [x \neq 0 \Longrightarrow x^2 > 0]$.

To compare these paragraph proofs and to see more clearly the logic involved, let's expand each of the sentences into a column. We begin with sentences (1a) and (1b).

Sentence (1a):

(1)	$\forall_x \forall_y \, [x \neq y \Longrightarrow (x > y \text{ or } y > x)]$		[Theorem 86a]
(2)	$a \neq 0 \Longrightarrow (a > 0 \text{ or } 0 > a)$		[(1)]
(3)	$a > 0 \text{ or } 0 > a$	[$a \neq 0$]	[(2)]

Sentence (1b):

(1)	$\forall_x \forall_y \, [x \neq y \Longrightarrow (x > y \text{ or } y > x)]$	[Theorem 86a]
(2)	$a \neq 0 \Longrightarrow (a > 0 \text{ or } 0 > a)$	[(1)]
(3)	$a \neq 0$	[assumption]
(4)	$a > 0 \text{ or } 0 > a$	[(2), (3)]

In preparing to prove the (a)-form of the theorem, one introduces a restriction, '$a \neq 0$', which will, at the end, be honored by accepting a restricted quantifier, '$\forall_{x \neq 0}$'. In the paragraph proof one says 'For $a \neq 0$,'.

In preparing to prove the (b)-form, one introduces an assumption, '$a \neq 0$', which will, later, be discharged. In the paragraph proof one says 'Suppose that $a \neq 0$.'.

If he wished, instead, to prove (a), he could replace (6) by:

(6′)	$a^2 > 0$	$[a \neq 0]$	[(5)]

The intent, here, is to indicate that, subject to the restriction '$a \neq 0$', (6′) follows from (5). The final step would be:

(7′)	$\forall_{x \neq 0}\ x^2 > 0$		[(1) - (5), (6′)]

The restriction on (6′) is a notice that in generalizing on 'a' one must use a similarly restricted quantifier. The justification for step (6′) is modus ponens and the rule (R_1) given in the COMMENTARY for page 7-36.

Here is another variant of the proof of (a):

(1)	$\forall_x\ [x \neq 0 \Longrightarrow x^2 \in P]$		[Exercise 7]
(2)	$a \neq 0 \Longrightarrow a^2 \in P$		[(1)]
(3)	$a^2 \in P$	$[a \neq 0]$	[(2)]
(4)	$\forall_x\ [x > 0 \Longleftrightarrow x \in P]$		[Theorem 83]
(5)	$a^2 > 0 \Longleftrightarrow a^2 \in P$		[(4)]
(6)	$a^2 > 0$		[(3), (5)]
(7)	$\forall_{x \neq 0}\ x^2 > 0$		[(1) - (6)]

*

A student who did not think to use Exercise 7 might give a paragraph proof of (a) based on the multiplication transformation principle for inequations [the sentences in the paragraph are numbered for reference purposes]:

(1a) For $a \neq 0$, it follows, by Theorem 86a, that either $a > 0$ or $0 > a$.

(2a) Suppose that $a > 0$. Then, by part <u>a</u> of the mtpi, $a^2 > 0$.

(3a) Suppose that $0 > a$. Then, by part <u>b</u> of the mtpi, $0 < a^2$--that is, $a^2 > 0$.

(4a) Since, in either case, $a^2 > 0$, it follows that, for $a \neq 0$, $a^2 > 0$. Consequently, $\forall_{x \neq 0}\ x^2 > 0$.

3. By Theorem 85,

$$ab < 0 \Longleftrightarrow -(ab) > 0.$$

Since $-(ab) = a \cdot -b$, it follows that

$$ab < 0 \Longleftrightarrow a \cdot -b > 0.$$

By part a of the ftpi,

$$a \cdot -b > 0 \Longleftrightarrow [(a > 0 \text{ and } -b > 0) \text{ or } (a < 0 \text{ and } -b < 0)].$$

By Theorem 85, $-b > 0$ if and only if $b < 0$ and $-b < 0$ if and only if $--b > 0$, that is, if and only if $b > 0$. Hence,

$$a \cdot b < 0 \Longleftrightarrow [(a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0)].$$

Consequently, $\forall_x \ldots$.

*

Answers for Part B.

[Students should interpret their answers with reference to the graph.]

1. The theorem in question may be stated as either:

(a) $\forall_{x \neq 0}\; x^2 > 0$ [for each nonzero x, $x^2 > 0$]

or:

(b) $\forall_x\; [x \neq 0 \Longrightarrow x^2 > 0]$ [for each x, if x is not 0 then $x^2 > 0$]

[Recall that in some cases a restricted quantifier is required if one is to avoid nonsense--'$\forall_{x \neq 0}\; x/x = 1$' is a case in point. However, the use of the restricted quantifier in (a) serves merely as an abbreviation. Similarly, merely abbreviatory uses of restricted quantifiers occur in the two parts of the multiplication transformation principle for inequations.] A theorem very much like (b) occurs in Exercise 7 of Part D on page 7-27. A student who recalls this exercise might prove (b) as follows:

(1)	$\forall_x\; [x \neq 0 \Longrightarrow x^2 \in P]$	[Exercise 7]
(2)	$a \neq 0 \Longrightarrow a^2 \in P$	[(1)]
(3)	$\forall_x\; [x > 0 \Longleftrightarrow x \in P]$	[Theorem 83]
(4)	$a^2 > 0 \Longleftrightarrow a^2 \in P$	[(3)]
(5)	$a \neq 0 \Longrightarrow a^2 > 0$	[(2), (4)]
(6)	$\forall_x\; [x \neq 0 \Longrightarrow x^2 > 0]$	[(1) - (5)]

Answers for Part A.

1. By Theorem 86e, if $(a > 0$ and $b > 0)$ then $ab > a0 = 0$.

   By Theorem 85, $a < 0$ and $b < 0$ if and only if $-a > 0$ and $-b > 0$. By the theorem just proved, if $-a > 0$ and $-b > 0$ then $-a \cdot -b > 0$. Hence, if $a < 0$ and $b < 0$ then $-a \cdot -b > 0$. But, by Theorem 23, $-a \cdot -b = ab$. Consequently, if $a < 0$ and $b < 0$ then $ab > 0$.

   Now, assuming that $(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$, it follows [by the dilemma--COMMENTARY for page 7-22] from the two results just obtained that $ab > 0$. Hence, if $[(a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)]$ then $ab > 0$. Consequently, $\forall_x$ ... .

2. [The reference to Theorem 81 in the third line of this exercise should, strictly, be supplemented by a reference to Theorem 83:

   Suppose that $ab > 0$. It follows, by Theorem 83, that $ab \in P$ and, by Theorem 81, that $ab \neq 0$.

   We shall often, as in the text, not bother to cite Theorem 83, and--subject to your discretion--students should have the same privilege.

   Continuing with the first paragraph of the proof in the text, one might say:

   Now, by the principle for multiplying by 0, if $b = 0$ then $ab = 0$. So, since $ab \neq 0$, it follows [by modus tollens] that $b \neq 0$. Hence, by Theorem 86a, ...]

   Here is a model for what students should write to fill the gap indicated in Exercise 2:

   Suppose that $b < 0$. By part <u>b</u> of the mtpi, it follows that

   $$0b < ab \text{ only if } 0 > a.$$

   Since $0b = 0$, it follows that, for $b < 0$,

   $$ab > 0 \text{ only if } a < 0.$$

   Since $ab > 0$, it follows that $a < 0$. So, for $b < 0$, we have

   $$\text{both } a < 0 \text{ and } b < 0.$$

*

The inference beginning the last paragraph of Exercise 2 is an example of the complex dilemma [see the COMMENTARY for page 7-34].

*

be better to replace the first sentence of the paragraph proof by:

> By Theorem 87, for $a > 0$, $a \neq 0$. So, for $a > 0$, by the mtpi, $a + \frac{1}{a} > 2$ if and only if $a^2 + 1 > 2a$.

Incidentally, students should not be expected to remember that '$\forall_x\, x \not> x$' is Theorem 87. Instead of 'By Theorem 87', a student might write 'By an earlier theorem'. He should, then, be prepared to state the theorem he has in mind. Moreover, it is not even important that the theorem he has in mind be a theorem which actually has already been proved. If, instead of Theorem 87, he quotes '$\forall_x \forall_y\, [x > y \Longrightarrow x \neq y]$', this is fine. The important thing is that he has thought of a theorem which he sees to be relevant and feels sure he could derive from theorems which have actually been proved. Instead of 'Theorem 87', a student might say '0 isn't positive'. From this [Theorem 81] and Theorem 83 he should be able to argue that, for $a > 0$, $a \neq 0$.

(4)	$(a + \frac{1}{a})a = a^2 + \frac{1}{a}a$	$[a \neq 0]$	[(3)]
(5)	$\forall_x \forall_{y \neq 0}\ (x \div y)y = x$		[pq]
(6)	$\frac{1}{a}a = 1$	$[a \neq 0]$	[(5)]
(7)	$(a + \frac{1}{a})a = a^2 + 1$		[(4), (6)]
(8)	$a^2 + 1 > 2a \Longleftrightarrow a + \frac{1}{a} > 2$		[(2), (7)]

Here, the first of the two restrictions on step (2) comes about from the fact that, by the rule for restricted universal instantiation, the auxiliary premiss '$a > 0$' is needed if one is to infer (2) from (1), and that, by (R_1), this premiss may be introduced as a restriction on (2), rather than as an assumption. The second restriction on (2), '$a \neq 0$', is required by (R_2), because (2) contains the expression '$1/a$'. The restriction on step (4) is also required by (R_2), again because (4) introduces '$1/a$' into the test-pattern. [The fact that '$1/a$' has previously entered the room through another door is irrelevant. It is tagged each time it enters.] The restriction on step (6) serves a dual purpose. Like the '$a > 0$' at step (2), it is an auxiliary premiss which is needed if one is to infer (6) from the restricted generalization (5). And, like the '$a \neq 0$'s at steps (2) and (4), it is demanded by (R_2).

Continuing the proof in column form would be tedious and serve no useful purpose. It is sufficient to note that the next-to-last step would be:

(n)	$a + \frac{1}{a} \geq 2$	[ ],

inferred from two earlier steps '$a \neq 1 \Longrightarrow a + \frac{1}{a} \geq 2$' and '$a = 1 \Longrightarrow a + \frac{1}{a} \geq 2$', the rule of reasoning involved being proof by cases [see Unit 6 COMMENTARY, TC[6-344]a], or the dilemma, using as a third premiss the logically valid sentence '$a = 1$ or $a \neq 1$'. The test-pattern completed by this next-to-last step is subject to the two restrictions '$a > 0$' and '$a \neq 0$'. Since, by Theorem 87, the first of these implies the second, one can disregard the latter and state the conclusion:

(n + 1)	$\forall_{x > 0}\ x + \frac{1}{x} \geq 2$	[(1) - (n), Th. 87]

In order to suggest the need for using (R_2) [and Theorem 87] it would

nonabbreviatory use of restricted quantifiers], we need a prescriptive rule, (R_2), for the introduction of restrictions in test-patterns.

For a complete formalization of, say, algebra one needs to formalize not only the rules of reasoning but also the "rules of grammar". That is, one needs, for example, rules for constructing words and sentences. Among such rules which one might adopt are:

> Variables and numerals are words and if W_1 and W_2 are words then so are $(W_1 + W_2)$, $(W_1 \cdot W_2)$, $-W_1$, $(W_1 - W_2)$, and $(W_1 \div W_2)$.
>
> If W_1 and W_2 are words then $W_1 = W_2$ and $W_1 > W_2$ are sentences.
>
> If S_1 and S_2 are sentences then so are $(S_1 \Longrightarrow S_2)$, not-$S_1$, ($S_1$ and S_2), and (S_1 or S_2).

In order to implement the use of the rule (R_2) we would need to supplement the definition of 'word' by a grammatical rule specifying the restriction associated with each word. This can be done in a fairly straightforward way, but, since we have, up to now, restricted our discussion of logic to the rules of reasoning, and have taken the rules of grammar for granted, we shall not pursue this matter further. Just as students learn to forgo treating an expression such as '$- \cdot 3 +$' as a word, so, they should have no difficulty in determining <u>what</u> restriction is needed at a step to which rule (R_2) applies. The only problem--and this, like many problems, is solved only by learning to pay attention to what one is doing--is to learn to recognize the situations to which (R_2) does apply. Just look at each assumption, and look at each step obtained by instantiation, to see whether there are substitutions which would lead to meaningless expressions. If there are, introduce a restriction which forbids these substitutions.

We can now profitably discuss the proof of Theorem 97c--specifically, the solution of Exercise 5 of Part E on page 7-40. We begin by exhibiting the first steps of a column proof for Theorem 97c. These steps constitute an expression into column form of the solution for Exercise 5 given earlier in this COMMENTARY in the answers for Part E.

(1) $\forall_x \forall_y \forall_{z>0} [xz > yz \Longleftrightarrow x > y]$ [mtpi]

(2) $(a + \frac{1}{a})a > 2a \Longleftrightarrow a + \frac{1}{a} > 2$ $[a > 0,\ a \neq 0]$ [(1)]

(3) $\forall_x \forall_y \forall_z\ (x + y)z = xz + yz$ [dpma]

Here, in order to derive (2), we need, by the principle of restricted universal instantiation, an auxiliary premiss '$a \neq 0$'. This might have been introduced into the test-pattern as an assumption but, by (R_1), may alternatively be introduced as a restriction on step (2). By (R_2), this same restriction must, in any event, be applied to step (2). So, in the above derivation the restriction plays a double role--first, to honor [through (R_1)] the restricted quantifier '$\forall_{y \neq 0}$' in (1) and, second, to prohibit substitutions which would result in meaningless expressions [(R_2)]. This double role is not surprising in view of the fact that, in the case of (1), the restricted quantifier was introduced to prohibit the formation of meaningless sentences as instances of (1). [One could, of course, have introduced '$a \neq 0$' as an assumption, to serve as an auxiliary, and also as a restriction, to satisfy (R_2). Doing so, one would obtain a different conclusion:

(1)	$\forall_x \forall_{y \neq 0} \frac{x}{y} \cdot y = x$		[basic principle]
(2)	$a \neq 0$		[assumption]*
(3)	$\frac{2}{a} \cdot a = 2$	[$a \neq 0$]	[(1), (2)]
(4)	$a \neq 0 \Longrightarrow \frac{2}{a} \cdot a = 2$		[(3); *(2)]
(5)	$\forall_{x \neq 0} [x \neq 0 \Longrightarrow \frac{2}{x} \cdot x = 2]$		[(1) - (4)]

Step (5) is, of course, a redundant way of saying what is said by '$\forall_{x \neq 0} \frac{2}{x} \cdot x = 2$'. Note that it is not necessary to repeat the restriction '[$a \neq 0$]' at step (4). Once step (3), at which '2/a' is introduced, is rendered acceptable by the restriction placed on it, (4) does follow from (3) by conditionalizing.]

To summarize the preceding discussion and that in the COMMENTARY for page 7-36, our use of restricted quantifiers requires us to make four additions to the rules of reasoning adopted in the Appendix of Unit 6. First, we need, in addition to the rule of universal instantiation, a rule of restricted universal instantiation. Second, we need to add to the statement of the test-pattern principle. Third, we need a permissive rule, (R_1), for the introduction of restrictions in test-patterns. [These three modifications are discussed in the COMMENTARY for page 7-36, and would be the only ones needed if restricted quantifiers served only an abbreviatory purpose.] Finally [in order to take account of our

for page 7-36, must now be completed to cover such nonabbreviatory uses of restricted quantifiers. The techniques of proof so far introduced need to be supplemented in such a way as to rule out such derivations as this one:

(1)	$\forall_x\, x \cdot 1 = x$	[basic principle]
(2)	$\frac{1}{a} \cdot 1 = \frac{1}{a}$	[(1)]
(3)	$\forall_x\, \frac{1}{x} \cdot 1 = \frac{1}{x}$	[(1), (2)]

The rules of reasoning adopted up to now would lead one to accept this as a proof of (3)--but, (3) is meaningless since it has the meaningless instance '$\frac{1}{0} \cdot 1 = \frac{1}{0}$'. What we need is a rule which will result in (3) being replaced by:

(3′) $\forall_{x \neq 0}\, \frac{1}{x} \cdot 1 = \frac{1}{x}$

This will result from the restatement of the test-pattern principle given in the COMMENTARY for page 7-36 if we adopt a rule which requires the introduction of the restriction '[$a \neq 0$]' on step (2). The introduction of this restriction at this point seems reasonable since it is here that the expression '$1/a$', which can be used to generate meaningless expressions, is introduced into the test-pattern. Since such an expression can be introduced into a test-pattern either by universal instantiation [as above], or by restricted universal instantiation, or in an assumption, our new rule will be stated to cover all three cases:

(R_2) A step which either is an assumption or results from universal instantiation [restricted or not] must be accompanied by a restriction forbidding substitution which would lead to the occurrence of meaningless expressions.

This rule, then, requires the introduction of the restriction '[$a \neq 0$]' on step (2), whereupon the test-pattern principle no longer justifies the conclusion (3), but does justify the desired conclusion (3′).

Before discussing Theorem 97c it will be helpful to take up another example of the application of rules (R_1) and (R_2):

(1)	$\forall_x \forall_{y \neq 0}\, \frac{x}{y} \cdot y = x$		[basic principle]
(2)	$\frac{2}{a} \cdot a = 2$	[$a \neq 0$]	[(1)]
(3)	$\forall_{x \neq 0}\, \frac{2}{x} \cdot x = 2$		[(1), (2)]

3. Yes, 2.

There is an approach to Exercise 3 which pulls in some of the ideas studied in Unit 5. Intuitively, we see that the low point of p is the point at which some horizontal line is tangent to the curve. Horizontal lines intersect the curve in two points, one point, or no points. So, to prove that p has a minimum, just find the number k such that the equation:

$$(*) \qquad x + \frac{1}{x} = k$$

has only one root. For $x \neq 0$, (*) is equivalent to:

$$x^2 - kx + 1 = 0$$

The discriminant of this equation is $k^2 - 4$, and for the equation to have precisely one root, $k^2 - 4$ must be 0. And this is the case if and only if $k = 2$ or $k = -2$. Since the low point is in the first quadrant, $k = 2$. [Interpret the finding '$k = -2$'.]

4. $\forall_{x > 0}\ x + \frac{1}{x} \geq 2$

5. [For a discussion of this solution, see below.] For $a > 0$, by the mtpi, $a + \frac{1}{a} > 2$ if and only if $a^2 + 1 > 2a$. The latter [$a^2 + 1^2 > 2a1$] is the case, by Theorem 97b, if $a \neq 1$. And, if $a = 1$ then $a + \frac{1}{a} = 2$. So, in any case, for $a > 0$, $a + \frac{1}{a} \geq 2$.

*

The theorem of Exercise 4--Theorem 97c:

$$\forall_{x > 0}\ x + \frac{1}{x} \geq 2$$

--is the first to be proved in this unit which contains a restricted quantifier which is not purely abbreviatory. In fact, the quantifier is partly abbreviatory and partly nonabbreviatory. The "abbreviatory part" could be eliminated by restating the theorem as:

$$\forall_{x \neq 0}\ [x > 0 \Longrightarrow x + \frac{1}{x} \geq 2]$$

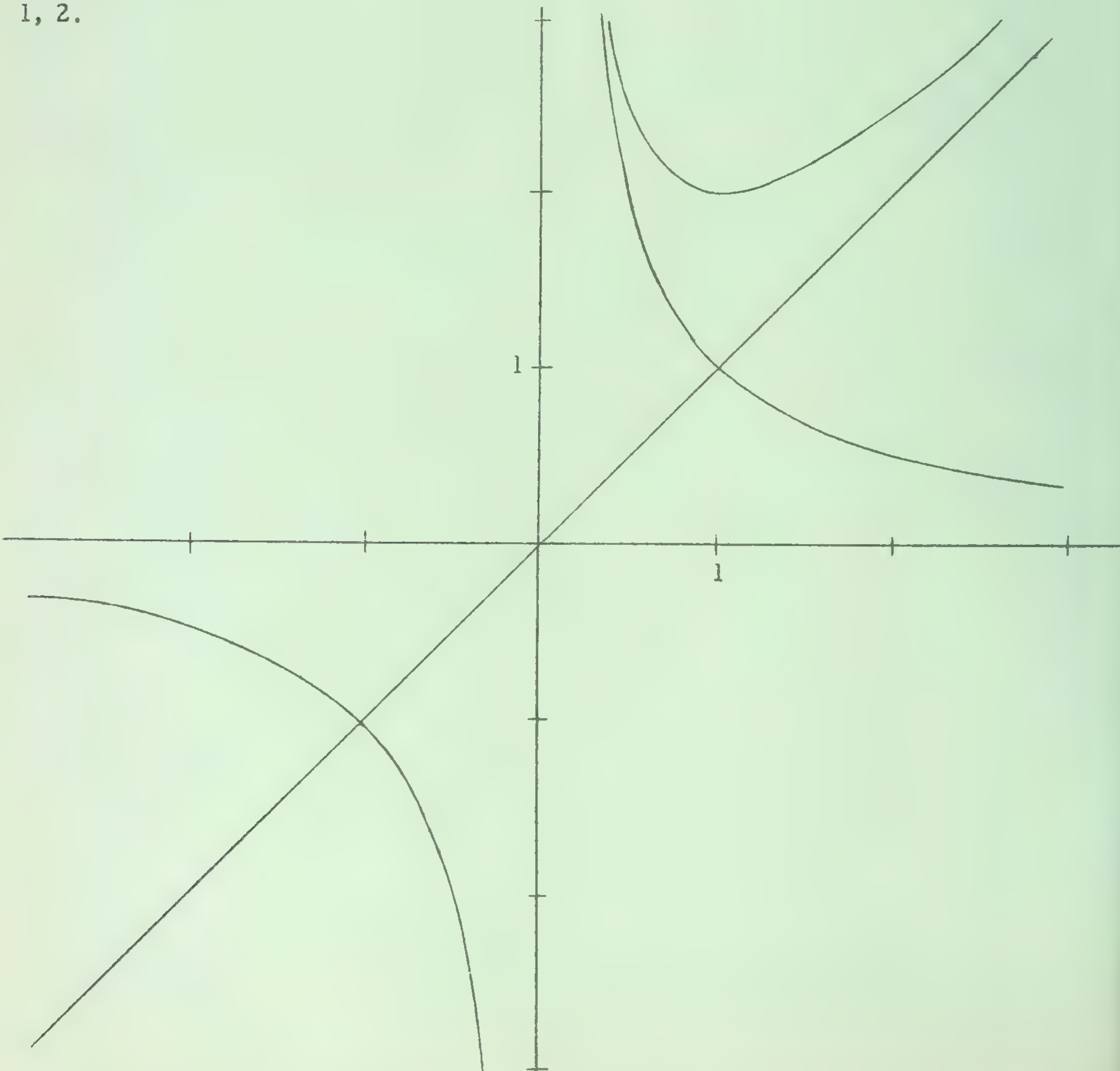
In this statement the restriction '$x \neq 0$' attached to the quantifier has, as its sole function, the prohibition of meaningless instances such as '$0 > 0 \Longrightarrow 0 + 1/0 \geq 2$' (meaningless because '$1/0$' is meaningless). The discussion of restricted generalizations, begun in the COMMENTARY

When assigning Part E, caution students to follow the instructions as to scale, and to make accurate graphs. You may need to remind them of the addition of ordinates method. It is described in the COMMENTARY for Part C on page 5-105 of Unit 5.

*

Answers for Part E.

1, 2.



function. The student is asked to graph the semi-perimeter function, p, in Part E, and to check his geometric intuition about the minimum value of p.]

1. $\frac{1}{2}$ and 2; 1 2. 1 3. 5 4. No

5. Yes, any point on the curve which has either coordinate ≥ 50 will do --for example, (1000, 1/1000) and (1/1000, 1000). [Your students may want to carry this further and find all points for which the perimeter is greater than 100. One sees intuitively that if he starts at a point for which the perimeter is at most 100 [say, at the point (1, 1)] and moves to the right along the curve, he will reach a point for which the perimeter is 100, and that, for each point to the right of this one, the perimeter will be greater than 100. Similarly, if one moves to the left along the curve, he will reach a second point for which the perimeter is 100, and, for each point to the left of this one, the perimeter will be greater. Incidentally, the set of points for which the perimeter is greater than 100 is, obviously, symmetric with respect to the 45°-line. Analytically, the problem of finding points for which the perimeter is greater than 100 amounts to finding positive solutions of the inequation '$2(x + 1/x) > 100$', and what has been intuited is that the equation '$2(x + 1/x) = 100$' has two roots--say r_1 and r_2, with $r_1 < r_2$--and that the solution set of the inequation is $\{x: 0 < x < r_1 \text{ or } x > r_2\}$. To check up on the correctness of this intuition it is sufficient to note, first, that, by the mtpi, the positive solutions of the inequation are exactly those of '$(x + 1/x)x > 50x$'-- that is, of '$x^2 - 50x + 1 > 0$' and, second, that, from the work in Unit 5 on quadratic functions, the solution set of this last inequation is $\{x: x < r_1 \text{ or } x > r_2\}$, when r_1 and r_2 are the roots of '$x^2 - 50x + 1 = 0$' ($r_1 < r_2$). It turns out that $r_1 = 25 - 4\sqrt{39} \doteq 0.02$ and $r_2 = 25 + 4\sqrt{39} \doteq 49.98$. As was to be expected from the intuitive argument, r_2 is a bit smaller than 50, and r_1 is the reciprocal of r_2. It is also interesting to note that the approximations given for r_1 and r_2 are both abscissas of points for which the perimeter is greater than 100. That (49.98, 1/49.98) is such a point follows from the fact that, since $49.98 < 50$, $1/49.98 > 1/50 = 0.02$ (see Exercise 3 of Part F on page 7-41). Hence, $49.98 + 1/49.98 > 49.98 + 0.02 = 50$. That (0.02, 1/0.02) is a point for which the perimeter is greater than 100 follows even more simply.]

6. The rectangle which has (1, 1) as one vertex has a minimum perimeter. [This is a consequence of Theorem 97c. See Exercise 5 of Part E on page 7-40.] There is none whose perimeter is a maximum.

*

Correction. On page 7-39, line 8 should be:

ure of the rhombus ? ---

Answers for Part C.

[These exercises are designed to promote discovery of an important inequality theorem through the use of geometric intuition. Imagine the geometric figure constructed out of sticks with pivots at D and C and at E and F. Hold ABCD fixed, and move $\overline{EF}$ up or down in the plane of ABCD. The maximum distance between $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ is CD. So, the maximum area-measure of the rhombus is the area-measure of the square. The maximum is attained only when the diagonals are congruent --that is, when the rhombus is a square. A theorem which tells us that the area-measure of the nonsquare rhombus is less than the area-measure of the square is:

$$\forall_{x > 0} \forall_{y > 0} [x \neq y \Longrightarrow x^2 + y^2 > 2xy]$$

It turns out, however, that a proof like that given below does not use the restrictions. Hence, a more general statement is possible. And this is Theorem 97b.]

1. $a^2 + b^2$; $2ab$

2. No; Yes

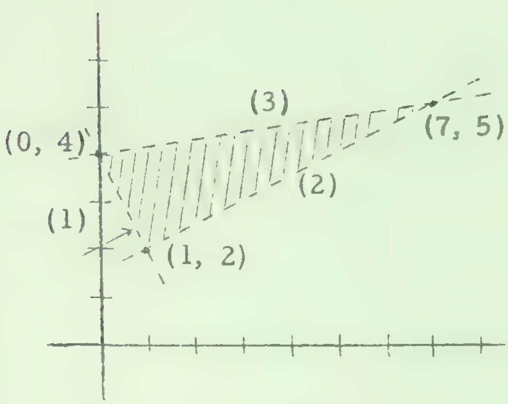
3. $\forall_x \forall_y [x \neq y \Longrightarrow x^2 + y^2 > 2xy]$; Suppose that $a \neq b$. Then, $a - b \neq 0$ and, by Theorem 97a, $(a - b)^2 > 0$. Since $(a - b)^2 = a^2 + b^2 - 2ab$, it follows, by Theorem 84, that $a^2 + b^2 > 2ab$. Consequently, $\forall_x \forall_y [x \neq y \Longrightarrow x^2 + y^2 > 2xy]$.

Answers for Part D.

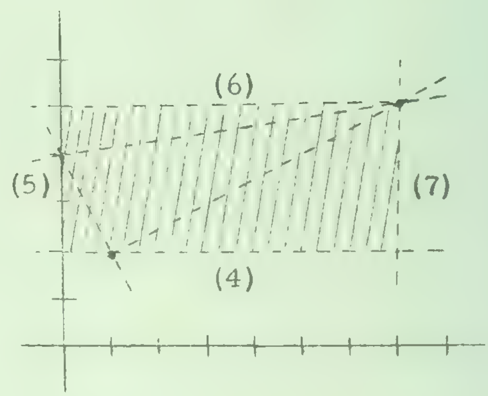
[Parts D and E use geometric intuition to promote discovery of another inequality theorem--Theorem 97c. It is surprising to discover that as a point moves along the graph, the corresponding rectangles have the same area-measure. But, it is geometrically obvious that the perimeter is not a constant function of the abscissa of the moving point. As one starts high on the curve (small abscissa) and moves down, the perimeter decreases and then increases (large abscissa). So, the perimeter appears to have a minimum value. Intuitive feelings about symmetry suggest that this minimum occurs when the point is on the graph of the identity

$$\begin{array}{lrcrcrcl}
(1) & 2x & + & y & - & 4 & > & 0 \\
(2') & -2x & + & 4y & - & 6 & > & 0 \\
\hline
 & & & 5y & - & 10 & > & 0 \\
(4) & & & & & y & > & 2
\end{array}
\qquad
\begin{array}{lrcrcrcl}
(1') & 14x & + & 7y & - & 28 & > & 0 \\
(3') & x & - & 7y & + & 28 & > & 0 \\
\hline
 & 15x & & & & & > & 0 \\
(5) & & & & & x & > & 0
\end{array}$$

$$\begin{array}{lrcrcrcl}
(2) & -x & + & 2y & - & 3 & > & 0 \\
(3) & x & - & 7y & + & 28 & > & 0 \\
\hline
 & & & -5y & + & 25 & > & 0 \\
(6) & & & & & y & < & 5
\end{array}
\qquad
\begin{array}{lrcrcrcl}
(2'') & -7x & + & 14y & - & 21 & > & 0 \\
(3'') & 2x & - & 14y & + & 56 & > & 0 \\
\hline
 & -5x & & & + & 35 & > & 0 \\
(7) & & & & & x & < & 7
\end{array}$$



Given system of inequations



Derived system of inequations

Since the solution set of the given system is a proper subset of the solution set of the derived system, the systems are not equivalent as in the case of systems of linear equations. So, for example, if one is looking for ordered pairs of integers which satisfy the given system, it is necessary to check [by substitution in the given system] each such ordered pair in the solution set of the derived system.]

*

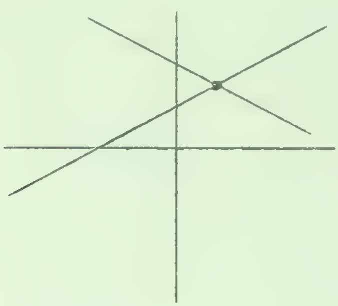
Answers for Part L.

1. No--a can of peaches costs more than 20¢. 2. 2¢; 3¢

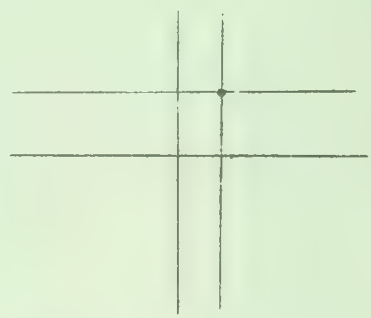
Answers for Part K.

1 (1, 3), (2, 3) 2. (1, 2) 3. (1, 2) 4. (2, 4), (3, 6), (4, 8)

[These exercises are easily solved by graphing. In fact, one of the purposes of Part J is to suggest this procedure to students. However, students may be curious about algebraic procedures especially in view of their experiences with systems of linear equations in Unit 5. There is a lot of interesting mathematics in analyzing the problems of solving systems of linear inequations, and this can provide good project work. In the case of a system of two linear equations, the general strategy is to derive from the given system a second system whose solution set not only contains the solution set of the given one but is also available by inspection of the equations in the system.



Given system of equations



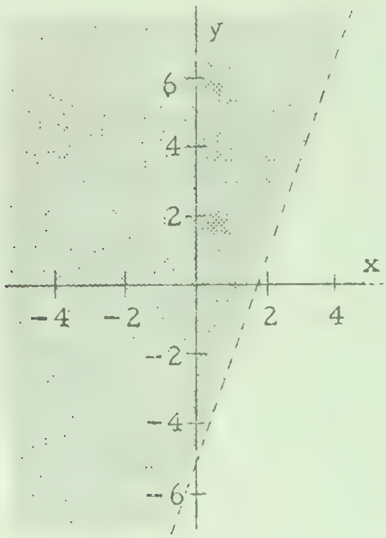
Derived system of equations

The same type of strategy is used in solving a system of linear inequations [in two variables]. If the solution set of the given system is the interior of a triangular region, one tries to derive a system whose solution set is the interior of the smallest rectangular region which contains the solution set of the given system and is also easily available by inspection. As an example, consider the following system:

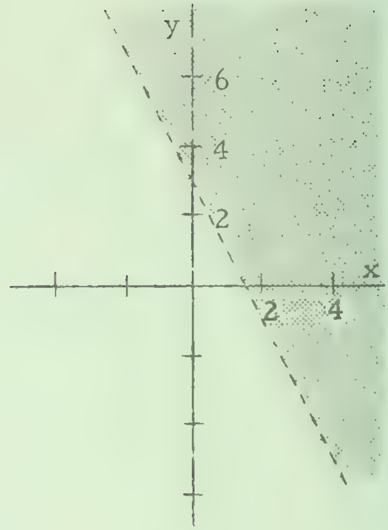
$$\begin{cases} (1)\ 2 + y - 4 > 0 \\ (2)\ -x + 2y - 3 > 0 \\ (3)\ x - 7y + 28 > 0 \end{cases}$$

Answers for Part J.

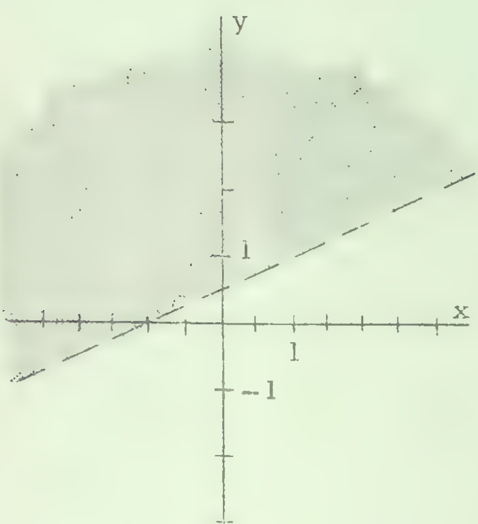
1.



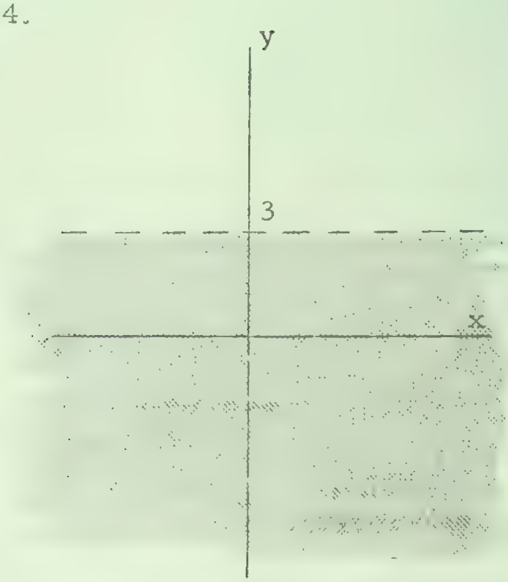
2.



3.



4.



Here are schematic justifications of such inferences:

$$\frac{\dfrac{(p \text{ or } q) \text{ and } r}{p \text{ or } q} \qquad \dfrac{\dfrac{\overset{*}{p} \qquad \dfrac{(p \text{ or } q) \text{ and } r}{r}}{p \text{ and } r}}{p \Rightarrow (p \text{ and } r)}{}^{*} \qquad \dfrac{\dfrac{\overset{*}{q} \qquad \dfrac{(p \text{ or } q) \text{ and } r}{r}}{q \text{ and } r}}{q \Rightarrow (q \text{ and } r)}{}^{*}}{(p \text{ and } r) \text{ or } (q \text{ and } r)}$$

$$\frac{(p \text{ and } r) \text{ or } (q \text{ and } r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{p \text{ and } r}}{p}}{p \text{ or } q} \qquad \dfrac{\overset{*}{p \text{ and } r}}{r}}{(p \text{ or } q) \text{ and } r}}{(p \text{ and } r) \Rightarrow [(p \text{ or } q) \text{ and } r]}{}^{*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overset{*}{q \text{ and } r}}{q}}{p \text{ or } q} \qquad \dfrac{\overset{*}{q \text{ and } r}}{r}}{(p \text{ or } q) \text{ and } r}}{(q \text{ and } r) \Rightarrow [(p \text{ or } q) \text{ and } r]}{}^{*}}{(p \text{ or } q) \text{ and } r}$$

*

[Needless to say, the exercises of Part I are intended to give students opportunities to practice their abilities for solving inequations, rather than as exercises in logical analysis. Analyses such as those contained in the preceding remarks on Exercises 5, 8, and 13 are for your guidance in answering questions. You may wish to elicit an approximation to such analyses in class discussions, but should feel entirely satisfied with students who are able to present written work like that shown at the bottom of TC[7-41, 42]i and be able to draw the graphs.]

*

Exercise 15. See Exercise 2 of Part F on page 7-41.

Exercise ☆16. $\forall_x \; \dfrac{x^2 + 2}{\sqrt{x^2 + 1}} = \dfrac{x^2 + 1}{\sqrt{x^2 + 1}} + \dfrac{1}{\sqrt{x^2 + 1}} = \sqrt{x^2 + 1} + \dfrac{1}{\sqrt{x^2 + 1}}$.

So, since for each x, $\sqrt{x^2 + 1} > 0$, Theorem 97c tells us that the given inequation is satisfied by all real numbers.

Here, since $36 = 6^2$ and $6 \geq 0$, the first line implies the second by Theorem 98b [more conveniently written '$\forall_x \forall_{y \geq 0} [x^2 < y^2 \Longrightarrow -y < x < y]$']. To see that the second line implies the first [and that, hence, the two are equivalent] one uses Theorem 98c. Briefly, either $x^2 + 5x \geq 0$ or $x^2 + 5x \leq 0$. In the first case, assuming [second line] that $x^2 + 5x < 6$, it follows, by Theorem 98c, that $(x^2 + 5x)^2 < 6^2$. In the second case, assuming [second line] that $-6 < x^2 + 5x$, one has, using Theorem 94, that $-(x^2 + 5x) < --6$ and since, in this case, $-(x^2 + 5x) \geq 0$, it follows, again by Theorem 98c, that $[-(x^2 + 5x)]^2 < (--6)^2$. So, in either case, $(x^2 + 5x)^2 < 36$. [Students may wish to note on page 7-151 that, by the argument just given, the converse of Theorem 98b is a theorem.]

The equivalence of the second line [that is, of '$-6 < x^2 + 5x$ and $x^2 + 5x < 6$'] and the third line follows from the atpi [and theorems about oppositing and subtraction].

The third and fourth lines are equivalent by virtue of two applications of the substitution rule for equations, the equations in the respective applications being '$x^2 + 5x + 6 = (x + 2)(x + 3)$' and '$x^2 + 5x - 6 = (x + 6)(x - 1)$'.

The fourth and fifth lines are equivalent by virtue of two applications of the substitution rule for biconditional sentences. One of the required biconditional sentences is obtained by the procedures used in showing that the first and third lines of the work for Exercise 5 are equivalent. The other bears a similar analogy to the work in Exercise 8.

Finally, the equivalence of the fifth and sixth lines depends, in part, on the validity of inferences of the forms:

$$\frac{\text{(p or q) and r}}{\text{(p and r) or (q and r)}} \quad \text{and:} \quad \frac{\text{(p and r) or (q and r)}}{\text{(p or q) and r}}$$

Now, here is a way to deduce the denial of the first component of the third line--that is, a way to expand the:

$$\frac{\text{theorems}}{\vdots \atop \text{not p}}$$

in the scheme above.

(1)	$x > 3$ and $x < -2$	[assumption]*
(2)	$x > 3$	[(1)]
(3)	$3 > -2$	[theorem]
(4)	$x > 3$ and $3 > -2$	[(2), (3)]
(5)	$\forall_x \forall_y \forall_z\ [(x > y \text{ and } y > z) \Longrightarrow x > z]$	[Theorem 86c]
(6)	$x > 3$ and $3 > -2 \Rightarrow x > -2$	[(5)]
(7)	$x > -2$	[(4), (6)]
(8)	$-2 > x$	[(1)]
(9)	$x > -2$ and $-2 > x$	[(7), (8)]
(10)	$(x > 3 \text{ and } x < -2) \Longrightarrow (x > -2 \text{ and } -2 > x)$	[(9); *(1)]
(11)	$\forall_x \forall_y$ not $(x > y \text{ and } y > x)$	[Theorem 86b]
(12)	not $(x > -2 \text{ and } -2 > x)$	[(11)]
(13)	not $(x > 3 \text{ and } x < -2)$	[(10), (12)]

<u>Exercise 13.</u>

$$(x^2 + 5x)^2 < 36$$

$$-6 < x^2 + 5x < 6$$

$$x^2 + 5x + 6 > 0 \text{ and } x^2 + 5x - 6 < 0$$

$$(x + 2)(x + 3) > 0 \text{ and } (x + 6)(x - 1) < 0$$

$$(x < -3 \text{ or } x > -2) \text{ and } -6 < x < 1$$

$$-6 < x < -3 \quad \text{or} \quad -2 < x < 1$$

(6)	$\forall_x \forall_y \forall_z\, (x > y \text{ and } y > z) \Longrightarrow x > z$	[Theorem 86c]
(7)	$(x > 3 \text{ and } 3 > 0) \Longrightarrow x > 0$	[(6)]
(8)	$x > 3 \text{ and } 3 > 0$	[(4), (5)]
(9)	$x > 0$	[(8), (7)]
(10)	$x > 0 \text{ and } x > 3$	[(9), (4)]
(11)	$x > 3 \Longrightarrow (x > 0 \text{ and } x > 3)$	[(10); †(4)]
(12)	$(x > 0 \text{ and } x > 3) \Longleftrightarrow x > 3$	[(3), (11)]

Note that the only undischarged premisses in this proof are '$3 > 0$' and the transitivity theorem for $>$. And these two things are precisely what one "feels" as he translates from '$x > 0$ and $x > 3$' to '$x > 3$'.

The proof of the other biconditional proceeds along similar lines.

<u>Exercise 8.</u>

$$x^2 - x - 6 < 0$$

$$(x - 3)(x + 2) < 0$$

$$(x > 3 \text{ and } x < -2) \quad \text{or} \quad (x < 3 \text{ and } x > -2)$$

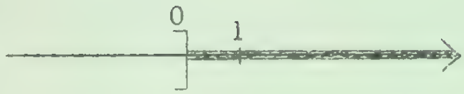
$$x < 3 \quad \text{and} \quad x > -2$$

In this problem, one might be curious about the justification of the equivalence of the third and fourth lines although the equivalence is intuitively clear from the fact that $3 > -2$ and Theorems 86b and c. [The fourth line is often abbreviated to '$-2 < x < 3$'.] That the third line is a consequence of the fourth is justified merely by one of the basic inference schemes for alternation sentences. But, that the fourth line is a consequence of the third requires a little more analysis. What one can do is deduce the denial of the first component of the third line--that is, deduce 'not ($x > 3$ and $x < -2$)'--and then apply the rule for denying an alternative and thus derive the fourth line. Schematically:

$$\boxed{\text{3rd line}} \qquad \qquad \boxed{\text{4th line}}$$

$$\dfrac{\dfrac{\dfrac{\overset{*}{p \text{ or } q} \qquad \overset{\underline{\text{theorems}}}{\vdots}\ \text{not } p}{q}}{(p \text{ or } q) \Longrightarrow q}\,* \qquad \dfrac{\dfrac{\overset{*}{q}}{p \text{ or } q}}{q \Longrightarrow (p \text{ or } q)}\,*}{(p \text{ or } q) \Longleftrightarrow q}$$

7.



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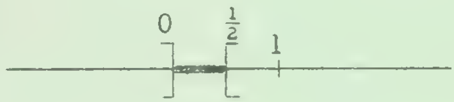

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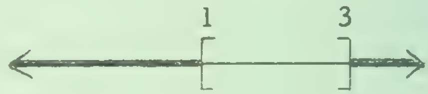

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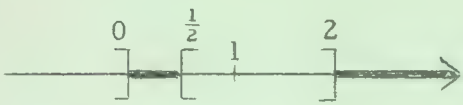

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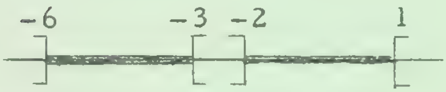

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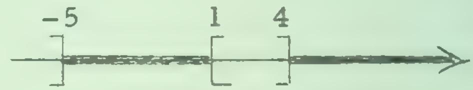


15. [Equivalent to Ex. 14]

☆16.

Exercise 5. Here is how the work leading up to the graph might appear:

$$x(x - 3) > 0$$

$$(x > 0 \text{ and } x > 3) \quad \text{or} \quad (x < 0 \text{ and } x < 3)$$

$$x > 3 \quad \text{or} \quad x < 0$$

Your students might be interested in a justification of the equivalence of these second and third lines. One part of the justification is provided by the substitution rule for biconditionals. The other part--how the biconditionals get there in the first place--is more interesting. The biconditionals are:

$(x > 0 \text{ and } x > 3) \Longleftrightarrow x > 3$ and: $(x < 0 \text{ and } x < 3) \Longleftrightarrow x < 0$

Here is a detailed proof of the first of these:

(1)	$x > 0$ and $x > 3$	[assumption]*
(2)	$x > 3$	[(1)]
(3)	$(x > 0 \text{ and } x > 3) \Rightarrow x > 3$	[(2); *(1)]
(4)	$x > 3$	[assumption]†
(5)	$3 > 0$	[theorem]

$$\forall_{x \in \mathcal{S}} \forall_{y \in \mathcal{S}} [x \neq y \Longrightarrow (x \text{ R } y \text{ or } y \text{ R } x)] \qquad \text{[cf. Th. 86a.]}$$

and:

$$\forall_{x \in \mathcal{R}} \forall_{y \in \mathcal{R}} \text{ not both } x \text{ S } y \text{ and } y \text{ S } x \qquad \text{[cf. Th. 86b.]}$$

Then, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathcal{S}$, from:

$$\forall_{x \in \mathcal{S}_1} \forall_{y \in \mathcal{S}_2} [x \text{ R } y \Longrightarrow f(x) \text{ S } f(y)]$$

one can derive:

$$\forall_{x \in \mathcal{S}_2} \forall_{y \in \mathcal{S}_1} [f(x) \text{ S } f(y) \Longrightarrow x \text{ R } y]$$

and:

$$\forall_{x \in \mathcal{S}_1 \cap \mathcal{S}_2} \forall_{y \in \mathcal{S}_1 \cap \mathcal{S}_2} [f(x) = f(y) \Longrightarrow x = y]$$

The derivations exactly parallel the solutions for Exercises 1 and 2. Aside from the minor replacements already mentioned, all one need do, essentially, is replace '>', when it occurs between 'f(a)' and 'f(b)', by 'S' and, when it occurs between 'a' and 'b', by 'R'. [When this results, for example, in 'b $\underline{\text{R}}$ a', read this as 'b R a or b = a'.] References to Theorems 88a and 87 should be interpreted as references to:

$$\forall_{x \in \mathcal{R}} \forall_{y \in \mathcal{R}} [(y \text{ R } x \text{ or } y = x) \Longrightarrow \text{not } (x \text{ R } y)]$$

and the corresponding analogue of Theorem 87, respectively.

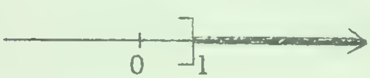
*

For examples like the exercises of Part I, see Unit 3, pages 3-103 through 3-105, and the accompanying COMMENTARY.

*

Answers for Part I [see notes following answers].

1.



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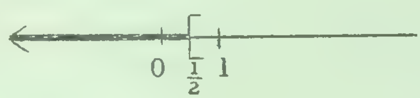

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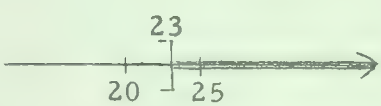

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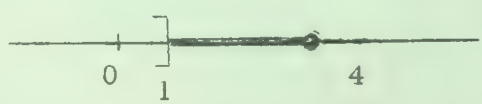

5.

6.

In fact, more generally, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are two sets of real numbers which are subsets of the domain of the real-valued function f, then

$$(\dagger\dagger) \qquad \forall_{x \in \mathcal{S}_1} \forall_{y \in \mathcal{S}_2} [x > y \Longrightarrow f(x) > f(y)]$$

implies:

$$\forall_{x \in \mathcal{S}_2} \forall_{y \in \mathcal{S}_1} [f(x) > f(y) \Longrightarrow x > y]$$

and:

$$\forall_{x \in \mathcal{S}_1 \cap \mathcal{S}_2} \forall_{y \in \mathcal{S}_1 \cap \mathcal{S}_2} [f(x) = f(y) \Longrightarrow x = y]$$

To see that this is the case, it is sufficient to review the solutions for Exercises 1 and 2. In re-reading the solution for Exercise 1, replace

'by (*),' by 'by ($\dagger\dagger$), for $b \in \mathcal{S}_1$ and $a \in \mathcal{S}_2$,',

'Therefore,' by 'Therefore, for $b \in \mathcal{S}_1$ and $a \in \mathcal{S}_2$,',

and '$\forall_x \forall_y$' by '$\forall_{x \in \mathcal{S}_2} \forall_{y \in \mathcal{S}_1}$'.

In re-reading the solution for Exercise 2, replace the sentence beginning 'By (*),' by:

By ($\dagger\dagger$), for $a \in \mathcal{S}_1$ and $b \in \mathcal{S}_2$, if $a > b$ then $f(a) > f(b)$, and, for $b \in \mathcal{S}_1$ and $a \in \mathcal{S}_2$, if $b > a$ then $f(b) > f(a)$.

Also, replace 'Therefore,' by 'Therefore, for $a \in \mathcal{S}_1 \cap \mathcal{S}_2$ and $b \in \mathcal{S}_1 \cap \mathcal{S}_2$,' and '$\forall_x \forall_y$' by '$\forall_{x \in \mathcal{S}_1 \cap \mathcal{S}_2} \forall_{y \in \mathcal{S}_1 \cap \mathcal{S}_2}$'.

A real insight into the relationships we have been investigating comes when one recalls the following facts:

(1) Theorems 88a and 87, which were used in the solutions, are immediate consequences of Theorem 86b. [For this, see page 7-33.]

(2) In the solutions, Theorem 86a [the only other theorem used] was applied only to arguments of f, and Theorems 88a and 87 only to values of f.

Now, suppose that f is a function with <u>any</u> domain $\mathcal{S}$ and <u>any</u> range $\mathcal{R}$, and suppose that there is a relation R among the members of $\mathcal{S}$ and a relation S among the members of $\mathcal{R}$ such that

2. Suppose that $a \neq b$. It follows, by Theorem 86a, that $a > b$ or $b > a$. By (*), if $a > b$ then $f(a) > f(b)$, and if $b > a$ then $f(b) > f(a)$. In either case, by Theorem 87, $f(a) \neq f(b)$. Therefore, if $f(a) = f(b)$ then $a = b$. Consequently, $\forall_x \forall_y\, [f(x) = f(y) \Longrightarrow x = y]$.

   [Note that the reference, above, to Theorem 87 can mean either of the two equivalent forms given on page 7-150. The second is more immediately applicable to the case in hand. For that matter, one might, instead, refer to Theorem 88a on page 7-33.]

*

From the preceding solutions of Exercises 1 and 2 it is easily seen that it is not necessary that the domain of f be the set of <u>all</u> real numbers. For, if the domain of f is any set $\mathcal{S}$ of real numbers then the same arguments show that:

$$(\dagger) \qquad \forall_{x \in \mathcal{S}} \forall_{y \in \mathcal{S}}\, [x > y \Longrightarrow f(x) > f(y)]$$

implies:

$$\forall_{x \in \mathcal{S}} \forall_{y \in \mathcal{S}}\, [f(x) > f(y) \Longrightarrow x > y]$$

and:

$$\forall_{x \in \mathcal{S}} \forall_{y \in \mathcal{S}}\, [f(x) = f(y) \Longrightarrow x = y]$$

[In words, an increasing function has an inverse which is also an increasing function.] The only changes needed in the given solutions are the replacement of 'by (*),' by 'by (†), for $a \in \mathcal{S}$ and $b \in \mathcal{S}$,', of 'Therefore,' by 'Therefore, for $a \in \mathcal{S}$ and $b \in \mathcal{S}$,', and of '$\forall_x \forall_y$' by '$\forall_{x \in \mathcal{S}} \forall_{y \in \mathcal{S}}$'.

The same arguments, analyzed a little more carefully, show that one can use:

$$\forall_x \forall_{y \geq 0}\, [x > y \Longrightarrow x^2 > y^2]$$

to prove:

$$\forall_{x \geq 0} \forall_y\, [x^2 > y^2 \Longrightarrow x > y]$$

and:

$$\forall_{x \geq 0} \forall_{y \geq 0}\, [x^2 = y^2 \Longrightarrow x = y]$$

[Compare with Theorem 98.]

So [for $b \neq 0$ and $c \neq 0$],

$$\frac{a}{c} > \frac{a}{b} \iff abc(b - c) > 0 .$$

So, for example, for $a > 0$, $b \neq 0$, and $c \neq 0$, by the ftpi,

$$\frac{a}{c} > \frac{a}{b} \iff [(bc > 0 \text{ and } b > c) \text{ or } (bc < 0 \text{ and } b < c)].$$

*

Answer for Part G.

For positive arguments, the larger the argument, the smaller the value --and the same for negative arguments. [In more technical language: The function r is <u>monotone</u> <u>decreasing</u> on the set P--and, also, on the set N.]

*

Answers for Part H.

1. Suppose that $f(a) > f(b)$. By Theorem 86a, $a > b$ or $b \geq a$. Suppose that $b \geq a$. If $b = a$ then $f(b) = f(a)$, and if $b > a$ then, by (*), $f(b) > f(a)$. So, if $b \geq a$ then $f(b) \geq f(a)$. But, by Theorem 88a [page 7-33], if $f(b) \geq f(a)$ then $f(a) \not> f(b)$. So, if $b \geq a$ then $f(a) \not> f(b)$. Since, by hypothesis, $f(a) > f(b)$, it follows that $b \not\geq a$. But, as previously noted, either $a > b$ or $b \geq a$. So, $a > b$. Therefore, if $f(a) > f(b)$ then $a > b$. Consequently, $\forall_x \forall_y \, [f(x) > f(y) \Longrightarrow x > y]$.

   [Compare this solution with the derivation, on page 7-34, of (**) from Theorem 86d. If f is the function such that, for each x, $f(x) = x + c$ then (*), in Part H, is Theorem 86d, and the generalization in Exercise 1 is (**).

   On the other hand, if, for some $c > 0$, $f(x) = xc$, for each x, then (*) becomes a consequence of Theorem 86e. So, by the result of Exercise 1, one can conclude from Theorem 86e [together with the theorems--Theorem 86a and Theorem 88a--used in the solution of Exercise 1] that, for $c > 0$, $\forall_x \forall_y \, [xc > yc \Longrightarrow x > y]$. From this one immediately derives:

   $$\forall_{z > 0} \forall_x \forall_y \, [xz > yz \Longrightarrow x > y],$$

   and this is equivalent to (***) on page 7-34.]

Hence [using the substitution principle for biconditional sentences],

$$b > c \Longleftrightarrow \frac{a}{c} > \frac{a}{b}.$$

Therefore, for $a > 0$, $b \neq 0$, and $c \neq 0$, if $bc > 0$ then $b > c$ if and only if $\frac{a}{c} > \frac{a}{b}$. Consequently, $\forall_{x > 0} \cdots$.

(c) By Theorem 87 [or Theorems 81 and 83], if $c > 0$ then $c \neq 0$. So, by the theorem of part (b), for $a > 0$, $b \neq 0$, and $c > 0$,

$$(*) \qquad bc > 0 \Longrightarrow \left(b > c \Longleftrightarrow \frac{a}{c} > \frac{a}{b}\right).$$

Suppose that $b > c$. Then, by Theorem 86e, for $c > 0$, $bc > c^2$. Also, by Theorem 97a, for $c > 0$, $c^2 > 0$. Hence, by Theorem 86c, $bc > 0$. So, by (*),

$$b > c \Longleftrightarrow \frac{a}{c} > \frac{a}{b}.$$

In particular,

$$b > c \Longrightarrow \frac{a}{c} > \frac{a}{b},$$

and assuming that $b > c$, $\frac{a}{c} > \frac{a}{b}$. So, for $a > 0$, $b \neq 0$, and $c > 0$, if $b > c$ then $\frac{a}{c} > \frac{a}{b}$. Consequently, $\forall_{x > 0} \cdots$.

[Note that, as in the third proof of Theorem 97a (See COMMENTARY for Exercise 1 of Part B on page 7-38), we have had to use modus ponens and conditionalizing merely to get the desired conclusion out from under the assumption 'b > c'.]

*

The theorems in Exercise 3 of Part F can be proved by another method. The alternative procedure makes use of the theorem of Exercise 1. Briefly,

$$\frac{a}{c} > \frac{a}{b} \Longleftrightarrow \frac{a}{c} - \frac{a}{b} > 0,$$

$$\frac{a}{c} - \frac{a}{b} = \frac{a(b-c)}{bc},$$

and [Exercise 1 or 2] $\qquad \frac{a(b-c)}{bc} > 0 \Longleftrightarrow abc(b-c) > 0.$

Answers for Part F.

1. By Theorem 97a, for $c \neq 0$, $c^2 > 0$. So, by the mtpi, for $c \neq 0$,

$$\frac{a}{c} > \frac{b}{c} \iff \frac{a}{c} \cdot c^2 > \frac{b}{c} \cdot c^2.$$

[Complete as in example preceding Part F.]

2. By the theorem of Exercise 1 [Theorem 99a], for $b \neq 0$,

$$\frac{a}{b} > \frac{0}{b} \iff ab > 0b.$$

By Theorem 53, for $b \neq 0$, $\frac{0}{b} = 0$ and, by the cpm and the pm0, $0b = 0$. ...

3. (a) Suppose that $bc > 0$. By the mtpi, for $b \neq 0$ and $c \neq 0$,

$$\frac{1}{c}(bc) > \frac{1}{b}(bc) \iff \frac{1}{c} > \frac{1}{b}.$$

[By various theorems from Unit 2], for $b \neq 0$ and $c \neq 0$, $\frac{1}{c}(bc) = b$ and $\frac{1}{b}(bc) = c$. So,

$$b > c \iff \frac{1}{c} > \frac{1}{b}.$$

Hence, for $b \neq 0$ and $c \neq 0$, if $bc > 0$ then $b > c$ if and only if $\frac{1}{c} > \frac{1}{b}$. Consequently, $\forall_{y \neq 0}$ ... .

(b) Suppose that $bc > 0$. By the theorem of part (a), for $b \neq 0$ and $c \neq 0$, it follows that

$$b > c \iff \frac{1}{c} > \frac{1}{b}.$$

By the mtpi, for $a > 0$,

$$\frac{1}{c} > \frac{1}{b} \iff \frac{1}{c} \cdot a > \frac{1}{b} \cdot a.$$

By earlier theorems [Theorem 63 and the cpm], for $b \neq 0$ and $c \neq 0$, $\frac{1}{c} \cdot a = \frac{a}{c}$ and $\frac{1}{b} \cdot a = \frac{a}{b}$. So,

$$\frac{1}{c} > \frac{1}{b} \iff \frac{a}{c} > \frac{a}{b}.$$

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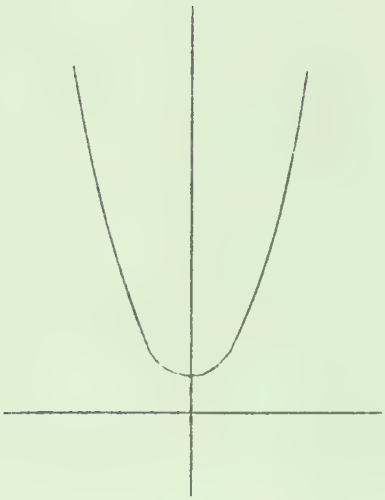

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Answer for Part C..

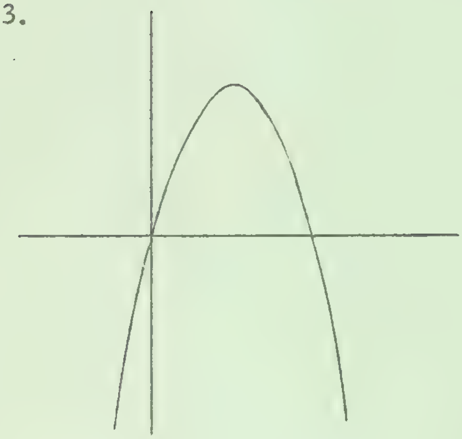
Equations 2, 4, and 5 have no roots.

*

Answers for Part D.

1. $-5, 5$
2. $0, -8$
3. $3, -2$
4. 3
5. $0, 1, -4$
6. $4, 3, -10$
7. -7
8. $2 + \sqrt{11},\ 2 - \sqrt{11}$

3.



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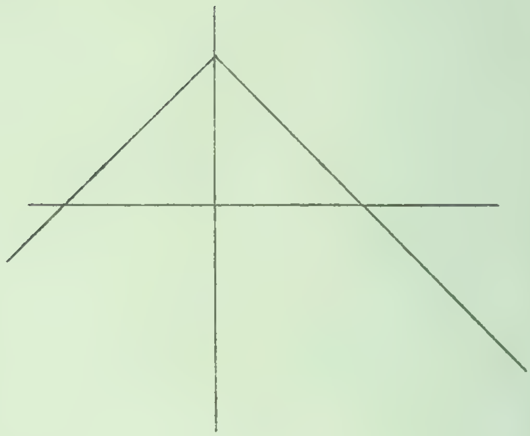

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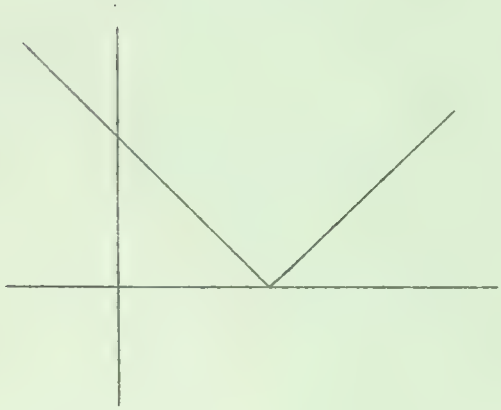

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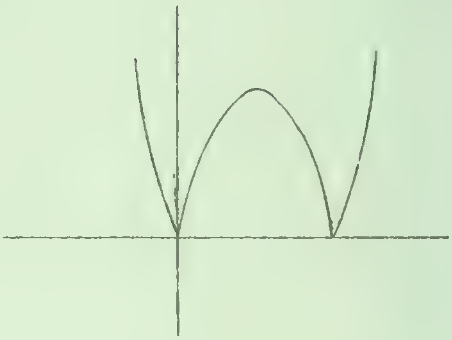

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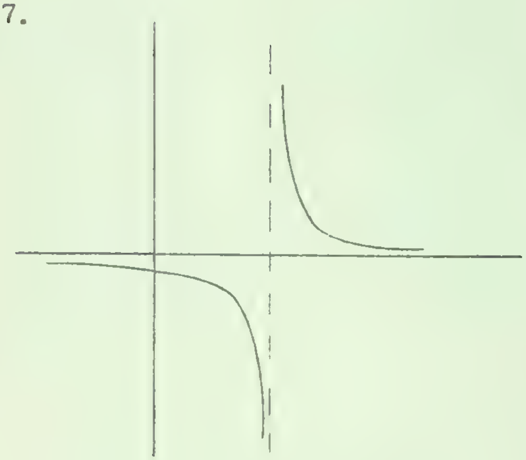

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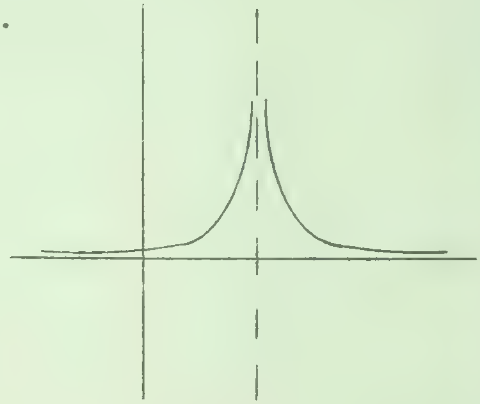


Answers for Miscellaneous Exercises.

[easy: A 1 - 18; C 1 - 5; D 1 - 8; E 1, 2, 6, 8; medium: B 1 - 9; E 3 - 5; hard: E 7, 9]

Answers for Part A.

1. $-\frac{1}{33}$ 2. $\frac{32}{33}$ 3. $\frac{4a + 2b}{a + b}$

4. $\frac{13x + 3y}{2(x + y)}$ 5. $\frac{3bc - 5ac - 7ab}{abc}$ 6. $\frac{9}{a - b}$

7. $\frac{k(x + y)}{(x - 2y)(2x - y)}$ 8. $\frac{k^2}{(x - 2y)(2x - y)}$ 9. $\frac{5x}{x + 2y}$

10. $9m$ 11. $\frac{x^3 - x^2 + 14x + 139}{(x + 5)(x + 4)}$ 12. $\frac{81m^2b}{(b + 1)^2}$

13. $3x + 12$ 14. $\frac{ab^2 - b + 2}{b^2}$

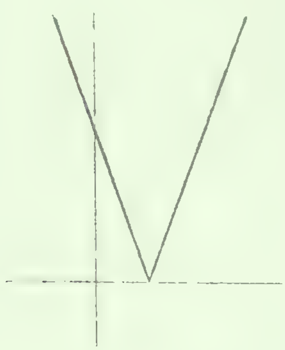
15. $\frac{k^3 - 1}{k(k - 1)}$ $\left[\text{or: } \frac{k^2 + k + 1}{k}\right]$ 16. $\frac{x^2 - 4x + 3 - 3xy - x^2y}{9 - x^2}$

17. $\frac{3t}{t^2 - 9}$ 18. $-\frac{t}{t + 3}$

*

Answers for Part B.

1.



2.

Answers for Part E.

1. Suppose that $(0, a)$ is the point. Then, $\sqrt{(0-4)^2 + (a-0)^2} = 2\sqrt{(0+1)^2 + (a-0)^2}$. So, $16 + a^2 = 4 + 4a^2$. Hence, $a = 2$ or $a = -2$. There are two such points, $(0, 2)$ and $(0, -2)$.

2. Suppose that (a, b) is the point. Then, $a = b$ and $\sqrt{(a-6)^2 + b^2} = \sqrt{(a-6)^2 + (b-8)^2}$. So, $b^2 = (b-8)^2$. Hence, $b = 4$. The required point is $(4, 4)$. [Alternatively, the required point is on the perpendicular bisector of $\overline{(6, 0)(6, 8)}$. Hence, it belongs to the locus of '$y = 4$'.]

3. Suppose that (x, y) is the point. Then, $\sqrt{(x-4)^2 + y^2} = \sqrt{x^2 + (y-4)^2}$. So, $x = y$. An equation of its path is '$y = x$'.

4. Suppose that (x, y) is the point. Then, $\sqrt{x^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$. So, $x^2 - 8x + 12 + y^2 = 0$. An equation of its path is '$(x-4)^2 + y^2 = 4$'. This is an equation of a circle with center $(4, 0)$ and radius 2.

5. Suppose that M is (x, y) and that P is (a, b). Then, $(a+8)/2 = x$ and $(b+0)/2 = y$; also, $a^2 + b^2 = 4$. So, $(2x-8)^2 + (2y)^2 = 4$. An equation of the path of M is '$(x-4)^2 + y^2 = 1$'. The path is a circle with center $(4, 0)$ and radius 1.

6. Since $\frac{1}{3} = 0.3333 + 0.0000\overline{3}$, $\left|\frac{1}{3} - 0.3333\right| = \frac{1}{30000}$.
Since $\frac{1}{7} = 0.142857 + 0.000000\overline{142857}$, $\left|\frac{1}{7} - 0.142857\right| = \frac{1}{7000000}$.

7. Suppose that the trip is d miles and that he travels at the speed limit for the entire trip. Then, his average speed is

$$\frac{d}{\frac{0.1d}{15} + \frac{0.9d}{50}}, \quad \text{or} \quad \frac{1500}{37} \text{ miles per hour.}$$

But, $1500/37 > 40$. So, he can average 40 miles per hour without exceeding the speed limits.

8. Equal. [Each fraction reduces to '$\frac{2}{3}$'.]

9. Suppose that his usual driving speed is x miles per hour. Then, his time for this unusual trip is $\frac{24}{x} + \frac{5}{6}$ hours. $\frac{1}{4}$ hour was spent in driving the first $\frac{x}{4}$ miles, $\frac{1}{12}$ hour was wasted in trying to fix the car by himself, $\frac{2}{x/2}$ hours were spent in driving 2 miles to the service station, and $\frac{3}{4}$ hour was spent in waiting for the car to be fixed. The rest of the time was spent in driving the remaining distance at the rate of $x + 12$ miles per hour. So,

$$\left(\frac{1}{4} + \frac{1}{12} + \frac{2}{x/2} + \frac{3}{4}\right) + \frac{24 - (\frac{x}{4} + 2)}{x + 12} = \frac{24}{x} + \frac{5}{6}.$$

That is, $x = 48$. Therefore, his usual driving speed is 48 miles per hour.

4. Yes. [Replace 'I^+' in the answer for Exercise 3 by 'S'. Step (4) is justified by the answer to Exercise 2.]

5.

(1)	$4 = 3 + 1$	[definition]
(2)	$\forall_x\, [x \in I^+ \Longrightarrow x + 1 \in I^+]$	[(ii)]
(3)	$3 \in I^+ \Longrightarrow 3 + 1 \in I^+$	[(2)]
(4)	$3 \in I^+$	[theorem (Exercise 3)]
(5)	$3 + 1 \in I^+$	[(3), (4)]
(6)	$4 \in I^+$	[(1), (5)]

Yes, '$4 \in S$' does follow from '$S \in \mathfrak{M}$' [and definition]. [Replace 'I^+', throughout the proof given above that $4 \in I^+$, by 'S'. Step (4) is now justified by the answer to Exercise 4.]

6. Students who believe, correctly, that each positive integer is either 1, or 1 + 1, or (1 + 1) + 1, or ((1 + 1) + 1) + 1, or ..., and who understand the systematic procedure illustrated by the column proof on page 7-46 and the column proofs asked for in Exercises 3 and 5, should answer 'no' to this question. Students who answer 'don't know' or 'yes' might be asked if they believe that there is a positive integer which one could never "count up to" [given enough time], starting with 1 and adding 1 at each count. If the answer is 'no' then, given any positive integer one could count off proofs, like that on page 7-46 and those of Exercises 3 and 5, until a proof was reached whose conclusion says that the given number belongs to I^+. [Students who object that no one has enough time to prove, in the manner suggested, that, say, $10^8 \in I^+$ (this would take at least 5000 sleepless years) should have it pointed out to them that 'Given lots of time!' is obviously intended to ward off such physiological considerations, and to invite fantasy.]

7. One who has answered 'no' to Exercise 6, and has understood the point made in Exercise 2 and 4, should grant that each positive integer belongs to each set S which is a member of $\mathfrak{M}$--that is, that I^+ is a subset of each such set S.

8. In view of what $\mathfrak{M}$ is [see Part B on page 7-45], (iii) says merely that I^+ is a subset of each set S which is a member of $\mathfrak{M}$. So, students who have given the expected answer to Exercise 7 should answer 'yes' for Exercise 8.

Answers for Part A.

1. Check marks in column (1) opposite (a), (b), (e), (f), (h), (i) and (j).

2. Check marks in column (2) opposite (a), (b), (c), (e), (f), (g), (h), (j), and (k).

3. (a), (b), (e), (f), (h), and (j)

4. [Various answers are possible--'the set of all integers greater than -2', 'the set of all positive integral multiples of $\frac{1}{3}$', 'the set of all real numbers greater than $\frac{1}{2}$', etc.]

*

Answers for Part B. [Note that $\mathcal{M}$ is a set of sets.]

1. [See answers for Exercises 3 and 4 of Part A.]

2. There are infinitely many sets which belong to $\mathcal{M}$.

3. [Students may or may not think so at this point. The purpose of Part C, which follows, is to convince students that I^+ is a subset of each member of $\mathcal{M}$.]

*

Answers for Part C.

1. Replace 'I^+', throughout the column proof given on page 7-46, by 'R'.

2. Yes. [Replace 'I^+' in the column proof on page 7-46 by 'S'. The result is a derivation of '$2 \in S$' from the definition '$2 = 1 + 1$' and the statements '$1 \in S$' and '$\forall_x [x \in S \Longrightarrow x + 1 \in S]$'. These last two statements follow from '$S \in \mathcal{M}$'. So, '$2 \in S$' is a consequence of the definition of '2' and '$S \in \mathcal{M}$'.]

3.

(1)	$3 = 2 + 1$	[definition]
(2)	$\forall_x [x \in I^+ \Longrightarrow x + 1 \in I^+]$	[(ii)]
(3)	$2 \in I^+ \Longrightarrow 2 + 1 \in I^+$	[(2)]
(4)	$2 \in I^+$	[theorem]
(5)	$2 + 1 \in I^+$	[(3), (4)]
(6)	$3 \in I^+$	[(1), (5)]

The purpose of the Exploration Exercises is to gain acceptance, on the part of students, of the statements (i), (ii), and (iii), given in Part C on page 7-46. These are, essentially, the basic principles (I^+_1), (I^+_2), and (I^+_3) for positive integers given on page 7-49. Statements (iii) and (I^+_3) are slightly different forms of the principle of mathematical induction.

As has been pointed out earlier in this COMMENTARY, basic principles are formulations of knowledge which the student already has. In the case of (I^+_1) - (I^+_3) this knowledge is

(*) that 1 is a positive integer,
that the sum of any positive integer and 1 is also a positive integer,
and that each positive integer is either 1, or 1 + 1,
or (1 + 1) + 1, or ((1 + 1) + 1) + 1, or ... .

The first two of these three propositions are easily formulated as (I^+_1) and (I^+_2). That the third proposition [including the '...'] is formulated precisely by (iii) is shown by the argument given in the two complete paragraphs on page 7-48. [This argument is mentioned in the COMMENTARY for page 7-23--see the bottom of TC[7-23]a--and is foreshadowed in Part C of the Exploration Exercises.] As shown in the COMMENTARY for page 7-49, (I^+_1), (I^+_2), and (iii), together, imply (I^+_3); and the latter implies (iii). Consequently, (I^+_1) - (I^+_3) constitute a precise formulation of the knowledge expressed in (*), above, and, so, completely characterize the subset I^+ of the set of real numbers.

It is on the basis of the knowledge expressed in (*) that students are expected to answer the Exploration Exercises and, in doing so, arrive at (i), (ii), and (iii).

*

When assigning the Exploration Exercises, call attention to the use of 'I^+', in (h) of Part A, as a name for the set of all positive integers. This notation was introduced in Unit 5.

Also, discuss possible answers to the question following (j) of Part A. [The question is largely rhetorical, its purpose being more to call attention to the '...' in (j) and in (k) than to bring forth precise answers.] '...' means that the set consists of all those numbers one would ever mention if he kept on extending the list "in the way it has been begun". In the case of (j), this means that the set consists of the halves of all the integers greater than or equal to −1. More briefly, '...' means the same as 'etc.' [but, what does 'etc.' mean?].

*

On the other hand, it follows from only the definition of K that

$$1 \in K \text{ and } \forall_{x \in K}\; x + 1 \in K.$$

So, (*) and the definition of K together imply '$I^+ \subseteq K$'.

In sum, in view of the way K is defined, (*) and '$I^+ \subseteq K$' are equivalent. Consequently, (*) is the sought-for third principle.

*

As one continues in his study of mathematics he is almost certain to come upon concepts and proofs which, after a sincere attempt at understanding, he does not fully grasp. In such a situation, it is important to realize that it is probably better to proceed on the basis of the limited understanding one has gained, resolving to return to the struggle after seeing what comes later, than to attempt, then and there, to fight through to complete mastery of the subject.

The discussion on page 7-48 may place many of your students in such a situation. They [and you] should not be discouraged if this happens. Solving the Exploration Exercises and a couple of careful readings of the text are sufficient for the present. Advise students who are still vague about page 7-48 to return to it from time to time after later experiences have paved the way for understanding.

*

As pointed out in the preceding COMMENTARY [as well as in the text], the argument on page 7-48 shows that it follows from (a), (b), and (*) that I^+ is precisely the set K of all those real numbers whose membership in I^+ follows from (a) and (b). Some of your students may be interested in the following slight variation of the argument.

The statement displayed on the fifth line from the bottom of page 7-48, says no more nor less than that K is a subset of the intersection of all sets S which satisfy (1). Since K is, itself, one of these sets, it follows that the intersection of all of them is a subset of K. Consequently, K is precisely the intersection of all sets S which satisfy (1).

Now, (*) says no more nor less than that I^+ is a subset of the intersection of all sets S which satisfy (1). And, since (a) and (b) say, precisely, that I^+ is, itself, one of these sets, it follows from (a) and (b) that the intersection of all of them is a subset of I^+. Hence, (a), (b), and (*) say, precisely, that I^+ is the intersection of all sets S which satisfy (1).

Since, as the argument on page 7-48 [ending with the fifth line from the bottom of the page] shows, K is the same intersection, it follows that (a), (b), and (*) are, together, equivalent to '$I^+ = K$'.

> <u>Correction.</u> On page 7-48, line 15 should end:
>
> --- $x + 1 \in K$.
>
> and line 19 should end:
>
> --- $x + 1 \in S$.

Students may question the need for proving that the positive integers are positive. It may be helpful to pose a question attributed to Abraham Lincoln: If you call a tail 'a leg', how many legs has a dog? The answer is, of course, 'four'. <u>Calling</u> a tail 'a leg' doesn't make it one. Similarly, calling certain numbers 'positive integers' does not ensure that they <u>are</u> positive. To make certain that they are positive we must, first, be sure what numbers we are talking about--that is, we need basic principles for I^+--and, then, we must prove that these numbers do, indeed, belong to P.

[At this point, some students may realize that (i), (ii), and (iii) on page 7-46 might be used as basic principles for I^+ and that, if they are so used, '$I^+ \subseteq P$' follows from (iii) together with '$1 \in P$' [Theorem 82] and '$\forall_x [x \in P \Longrightarrow x + 1 \in P]$' [a consequence of Theorem 82 and (P_3)]. This is, in fact, how Theorem 101 is proved on page 7-49.]

*

Statements (1) - (7) are the only assumptions which were made about I^+ in earlier units. In this unit we shall derive the first six of them from our basic principles. Statement (1) is Theorem 101 on page 7-49; (2) combines Theorems 102 and 103 on page 7-56; (3) is Theorem 105 on page 7-84; (4) is Exercise 3 of Part B on page 7-86; (5) is Theorem 108 on page 7-88; and (6) is a consequence of Theorem 118e on page 7-105.

*

The discussion on page 7-48 has been foreshadowed in the Exploration Exercises. [See, in particular, TC[7-45, 46]a, and the discussion of Exercises 6, 7, and 8 of Part C on page 7-46.] Statement (*) on page 7-48 is, of course, merely an abbreviation of (iii) of Exercise 8 on page 7-46.

Using less precise language than that of the text, one might say that K is the set whose members are the numbers 1, 1 + 1, (1 + 1) + 1, ((1 + 1) + 1) + 1, etc. [The 'etc.' introduces an element of vagueness which is absent from the definition of K given in the text.] It follows from the definition of K, in terms of (a) and (b), that each member of K is a positive integer--that is, that $K \subseteq I^+$. What we want is a third principle--it turns out to be (*)--which will tell us that $I^+ \subseteq K$. To find such a principle we begin by noting that from only the definition of K, it follows that

$$\forall_S [(1 \in S \text{ and } \forall_{x \in S}\, x + 1 \in S) \Longrightarrow K \subseteq S].$$

[The argument given on page 7-48 to show that this is the case has been foreshadowed in Exercises 2 and 4 of Part C on page 7-46, and resembles the argument given on page 7-23 to show that Theorem 82 cannot be derived from our first fourteen basic principles.] So, the statement '$I^+ \subseteq K$' and the definition of K together imply:

$$(*) \qquad \forall_S [(1 \in S \text{ and } \forall_{x \in S}\, x + 1 \in S) \Longrightarrow I^+ \subseteq S]$$

[For the numbers of arithmetic, the analog of positiveness is the property of being nonzero--a moderately uninteresting property--and the role of theorems on opposition and subtraction is played by theorems on symmetric differencing.]

In organizing the theory of numbers of arithmetic deductively, the role played in the theory of real numbers by the basic principles (I^+_1) - (I^+_3) is, here, taken over by three principles for whole numbers:

(W_1) $\quad 0 \in W$

(W_2) $\quad \forall_n\ n + 1 \in W$

(W_3) $\quad \forall_S\ [(0 \in S \text{ and } \forall_n\ [n \in S \Longrightarrow n + 1 \in S]) \Longrightarrow \forall_n\ n \in S]$

[In these, the domain of 'n' is the set W of whole numbers, and that of S is the set of subsets of the set of numbers of arithmetic.]

In addition to these, a completeness principle is needed but, as in the case of the real numbers, pending the introduction of such a principle much can be done with a cofinality principle:

$$\forall_x \exists_n\ n \geq x$$

On the basis of the first 12 of these 17 basic principles for numbers of arithmetic, and of suitable definitions of 'real number' and of the operations of addition, multiplication, opposition, subtraction, and division of real numbers, it is possible to derive the 11 basic principles for real numbers given on page 7-145. [The definition of "real number" is suggested in the COMMENTARY for page 1-1 of Unit 1.] Adding a definition of 'P' one can derive the principles (P_1) - (P_4). [Alternatively, adding a definition of the relation > for real numbers, one can prove Theorem 86. In either case the 13th basic principle, (G_a), for numbers of arithmetic would be a convenience in carrying out the details, but, since it is merely a contextual definition, it could be dispensed with.] Finally, adding the last four basic principles for numbers of arithmetic and a suitable definition of 'I^+' would make possible the proof of (I^+_1) - (I^+_3) and the cofinality principle for real numbers.

[A large part of the program outlined in the preceding paragraph was carried out in some detail in the real numbers unit of the 1958 UICSM Third Course.]

basic principles for real numbers which are given on page 7-145, all but two--the principle of opposites and the principle for subtraction--hold for numbers of arithmetic. [More precisely, the principles in question remain true if the domain of the variables is taken to be the set of numbers of arithmetic--rather than, as on page 7-145, the set of real numbers, and the operations referred to are taken to be the operations of addition, multiplication, and division of numbers of arithmetic.] In place of the principle of opposites one must be satisfied with the cancellation principle for addition [using which one can, as on page 2-66 of Unit 2, prove the principle for multiplying by 0], and a principle peculiar to the numbers of arithmetic:

<u>The 0-sum principle</u>.

$$\forall_x \forall_y \, [x + y = 0 \Longrightarrow x = 0]$$

Instead of adopting the cancellation principle as a basic principle, it is more convenient to adopt two principles concerning the operation of <u>symmetric differencing</u>. Intuitively, the symmetric difference of two numbers of arithmetic is the number one must add to the smaller to obtain the larger [and the symmetric difference of a number and itself is 0]. Using 'Δ' to denote this operation we have two <u>principles for symmetric differencing</u>:

(Δ_1) $\quad \forall_x \forall_y \, (x + y) \Delta y = x$

(Δ_2) $\quad \forall_x \forall_y \, [x + (x \Delta y) = y \text{ or } y + (x \Delta y) = x]$

[The cancellation principle for addition is an immediate consequence of (Δ_1). For, if $a + c = b + c$ then $(a + c) \Delta c = (b + c) \Delta c$. So, if $a + c = b + c$ then, by (Δ_1), $a = b$.]

As a basis for the study of order properties of the numbers of arithmetic it is convenient to adopt a principle for $\geq$:

(G_a) $\quad \forall_x \forall_y \, [x \geq y \Longleftrightarrow x = y + (x \Delta y)]$

[Then, if desired, '>' can be defined as '$\geq$ but $\neq$'.]

So far, we have mentioned 13 basic principles for numbers of arithmetic--9 which are analogs of certain basic principles for real numbers, and 4 which are displayed above. From these one can derive analogs of more than half of the first 100 theorems on real numbers--all those, in fact, which do not refer to opposition, subtraction, or positiveness.

despite the fall of the first. To obtain such an example, let U be the set of real numbers whose members are

$$\ldots, \frac{n+3}{n+2}, \ldots, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}$$

and consider the set $S \cup U \cup T$ arranged, as usual, in the natural order. In this case, not only does each domino have an immediate successor [which it would knock down _if_ it fell] but each except the first _is_ some domino's immediate successor. Still, tipping over $\frac{1}{2}$ will only result in knocking down the members of S--assuming, as before, that the spacing is such that no domino after the first is knocked down by anything but its immediate predecessor. [Although each member of $U \cup T$ has an immediate predecessor, none has its immediate predecessor in S.]

The upshot of all this [as far as concerns (I^+_3) and dominoes] is that (I^+_3) tells us that the positive integers are arranged in the way an endless line of dominoes has to be arranged _if_ all are to fall down when the first does. And this is quite different from saying that (I^+_3) is plausible because all in an endless line of dominoes _will_ fall down when the first does.

*

On page 7-89 we shall introduce another basic principle which concerns positive integers. This, _the cofinality principle_, says that there are no real numbers which are greater than all positive integers--that is, that "the positive integers go out as far as the real numbers do". After the introduction, in a later unit, of a _completeness principle_, the cofinality principle will become a provable theorem.

*

On the system of numbers of arithmetic.--Now that we have adopted all but one of the really essential basic principles for real numbers [the exception is the completeness principle mentioned above], it may interest you to compare the system of real numbers with that of numbers of arithmetic. The following is an outline of such a comparison.

As far as concerns addition, multiplication, and division, the numbers of arithmetic behave like the nonnegative real numbers and, of the eleven

[That the dominoes form an "endless line" is implied by (2) and, so, need not be stated as a third condition.] To describe some of the ways in which the dominoes might be arranged it will be simpler to think of them as real numbers, and to consider various sets of real numbers. To begin with consider the set S whose members are the real numbers

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{12}{12+1}, \dots$$

arranged in the natural order. This means that $\frac{1}{2}$ is the first member of S (first domino) and, for each n, $\frac{n+1}{n+2}$ is the one next succeeding $\frac{n}{n+1}$. In this case the ordering is that of the positive integers, and knocking down the first domino will ensure the fall of all, assuming that the spacing is such that each knocks down its successor.

To obtain a second example, let T be the set of all real numbers each of which is greater by 1 than some member of S. The members of T are, then, the numbers

$$\frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, \dots, \frac{2n+1}{n+1}, \dots.$$

Now, consider the set $S \cup T$, arranged in the natural [i.e., increasing] order [that is, with $\frac{1}{2}$ as first member, and, for each n, $\frac{n+1}{n+2}$ as the one next succeeding $\frac{1}{n+1}$, and $\frac{2n+3}{n+2}$ as the one next succeeding $\frac{2n+1}{n+1}$]. In this case, knocking down $\frac{1}{2}$ will ensure that all members of S fall over. But, assuming that each domino other than the first is knocked over, if at all, only by its immediate predecessor, there is nothing which will knock over $\frac{3}{2}$. Hence, all members of T will be left standing.

The difficulty in the second example arises from the fact that there is a domino $[\frac{3}{2}]$ other than the first which is not the one next following any others. Such a domino has nothing to knock it down. However, it is not difficult to give an example in which each domino other than the first does have an immediate predecessor and, yet, many remain standing

The second answer, above, pointing out the need for the principle of mathematical induction, is illuminated by the following analogy: Suppose that you are acquainted with three boys in the Smith family, Tom, Dick, and Harry, and know that each of them has red hair. In order to infer that each boy in the Smith family has red hair, you need an additional premiss: Each boy in the Smith family is either Tom, Dick, or Harry.

In the same way, even if one overlooks the vagueness inherent in the 'etc.', he is not justified in inferring '$\forall_n\ n \in S$' from:

$$1 \in S,\ 1 + 1 \in S,\ (1 + 1) + 1 \in S,\ ((1 + 1) + 1) + 1 \in S,\ \text{etc.}$$

In addition he needs a premiss:

$$\forall_n\ [n = 1 \text{ or } n = 1 + 1 \text{ or } n = (1 + 1) + 1 \text{ or } \ldots\ ]$$

The statement (*) on page 7-48 says precisely what this premiss is intended to say, without the vagueness of the '...'.

*

Students are often asked to accept (I^+_3), or an equivalent principle, on the ground of an analogy with a line of dominoes all of which tip over, in succession, once the first one is pushed. Here is a paraphrase of one of the more careful expositions of this kind:

> The principle (I^+_3) becomes plausible if one thinks of an endless line of standing dominoes. They will all be tipped over if, first, the first domino is tipped over and, second, the dominoes are so spaced that any one which is tipped over tips over the next.

That the first of these two sentences is true is unfortunate in view of the fact that the second is false. Even if, as is only fair, one assumes the existence of a set of dominoes arranged as described, it does not follow that tipping over the first ensures that all the others will fall down. There are [infinitely] many ways in which the dominoes might be arranged so as to satisfy the given conditions, and in <u>all</u> <u>but</u> <u>one</u> of these some dominoes will remain standing. [The principle (I^+_3) says, in effect, that the positive integers are arranged in this one exceptional way.]

The conditions to be satisfied are that

(1) there is a first domino, and

(2) for each domino there is a next succeeding one [so situated that, in falling, each domino knocks over its successor].

In discussing mathematical induction one sometimes uses the language of <u>properties</u>, rather than, as is done in this unit, the language of sets. For convenience, one says that a property is <u>hereditary</u> if, whenever it holds of a positive integer n, it also holds of n + 1. The principle of mathematical induction [(I^+_3)] can then be stated as follows:

> Each positive integer has every hereditary property which holds of 1.

In fact, given any hereditary property which holds of 1, let S be the set of all objects which have this property. Since the property holds of 1, $1 \in S$. Since the property is hereditary, $\forall_n [n \in S \Longrightarrow n + 1 \in S]$. Hence, by [$(I^+_3)$], $\forall_n\ n \in S$--that is, each positive integer has the given property. [The preceding argument is easily reversed. Given any set S such that $1 \in S$ and $\forall_n [n \in S \Longrightarrow n + 1 \in S]$, consider the property of being a member of S. Clearly, the property holds of 1 and is hereditary. Hence, by the statement displayed above, $\forall_n\ n \in S$.]

We have chosen to use the language of sets, rather than the language of properties, because students are familiar with the former from their work in earlier units.

*

The need for the principle of mathematical induction--as a basic principle --is sometimes questioned on the following grounds:

Suppose (a) that $1 \in S$ and (b) that $\forall_n [n \in S \Longrightarrow n + 1 \in S]$. From (b) it follows that if $1 \in S$ then 1 + 1--that is, 2--$\in S$. So, by (a), $2 \in S$. From (b) it follows that if $2 \in S$ then 2 + 1--that is, 3--$\in S$. So, since $2 \in S$, it follows that $3 \in S$. From (b) it follows that if $3 \in S$ then 3 + 1--that is, 4 --$\in S$. So, since $3 \in S$, it follows that $4 \in S$. Etc. Since each positive integer will, eventually, be reached in this way, each positive integer belongs to S.

Those who question the need for the principle of mathematical induction play down the role of "[Since] each positive integer will, eventually, be reached in this way", and may claim that the infinite sequence of arguments suggested by 'Etc.' is a proof for '$\forall_n\ n \in S$'. To this view there are two answers. First, a proof should be of finite length--it should be possible to exhibit <u>all</u> steps of a proof. Second, even if one were to admit proofs of infinite length, one must still, in such a proof, present evidence that the proof does, indeed, cover all instances of the generalization '$\forall_n\ n \in S$'. This is, in fact, the role of the statement "each positive integer will, eventually, be reached in this way". And, as proved on page 7-48, this is precisely what (*) says.

It is not hard to see that (c) on page 7-49 implies (iii). In the first place, by definition, '$I^+ \subseteq S$' means the same as '$\forall_{x \in I^+} x \in S$'. So, the consequents of the conditionals in (iii) and (c) are freely interchangeable. In the second place, '$\forall_x [x \in S \Longrightarrow x + 1 \in S]$', in (iii), implies the restricted generalization '$\forall_{x \in I^+} [x \in S \Longrightarrow x + 1 \in S]$' in (c). So, the antecedent [say, p] of the conditional in (iii) implies the antecedent [say, q] of the conditional in (c). But, by the rule of the hypothetical syllogism [Exercise 6 on page 6-377 of Unit 6], the inference:

$$\frac{p \Longrightarrow q \qquad\qquad q \Longrightarrow \overset{\text{(c)}}{r}}{\underset{\text{(iii)}}{p \Longrightarrow r}}$$

is valid. [More briefly: The antecedent in (iii) is "stronger" than the antecedent in (c); and, by the rule of the hypothetical syllogism, of two conditionals which have the same consequent, the one with the "weaker" antecedent implies (or: is stronger than) the other.]

It is also the case that (a), (b), and (iii), together imply (c) [see below]. Consequently, (a), (b), and (iii) [equivalently: (a), (b), and (*)] are, together, equivalent to (a), (b), and (c).

To show that (a), (b), and (iii) yield (c), suppose that $1 \in S$ and that $\forall_{x \in I^+} [x \in S \Longrightarrow x + 1 \in S]$. Since, by (a), $1 \in I^+$, it follows that $1 \in I^+ \cap S$. Since, by (b), $\forall_{x \in I^+} x + 1 \in I^+$, it follows that $\forall_{x \in I^+} [x \in I^+ \cap S \Longrightarrow x + 1 \in I^+ \cap S]$. Hence, by (iii) [with '$I^+ \cap S$' in place of 'S'], it follows that $I^+ \subseteq I^+ \cap S$. So, $I^+ \subseteq S$--that is, $\forall_{x \in I^+} x \in S$. Consequently [assuming (a), (b), and (iii)], if $1 \in S$ and $\forall_{x \in I^+} [x \in S \Longrightarrow x + 1 \in S]$ then $\forall_{x \in I^+} x \in S$.

*

> Correction. On page 7-48, line 9b should read:
>
> So, --- the sentence '$\forall_{x \in I^+} x \in S$', we shall write

In discussing the proof of Theorem 101 it may be helpful to expand it, on the board, to a column proof.

(1)	$\forall_S [(1 \in S \text{ and } \forall_{x \in S}\ x + 1 \in S) \Longrightarrow I^+ \subseteq S]$		[(*)]
(2)	$(1 \in P \text{ and } \forall_{x \in P}\ x + 1 \in P) \Longrightarrow I^+ \subseteq P$		[(1)]
(3)	$1 \in P$		[Theorem 82]
(4)	$\forall_x \forall_y [(x \in P \text{ and } y \in P) \Longrightarrow x + y \in P]$		[(P_3)]
(5)	$(a \in P \text{ and } 1 \in P) \Longrightarrow a + 1 \in P$		[(4)]
(6)	$a \in P \text{ and } 1 \in P$	[$a \in P$]	[(3)]
(7)	$a + 1 \in P$		[(5), (6)]
(8)	$\forall_{x \in P}\ x + 1 \in P$		[(3) - (7)]
(9)	$1 \in P \text{ and } \forall_{x \in P}\ x + 1 \in P$		[(3), (8)]
(10)		$I^+ \subseteq P$	[(2), (9)]

If, in (*), the second component of the antecedent of the conditional had been '$\forall_x [x \in S \Longrightarrow x + 1 \in S]$' [that is, if (iii) on page 7-46 were used in place of (*)], the restriction on step (6), above, would have occurred, instead, as an assumption, say, (5.5). In this case, the comment for (6) would be '[(3), (5.5)]'. There would be an additional step:

(7.5)	$a \in P \Longrightarrow a + 1 \in P$	[(7); *(5.5)]

and step (8) would be '$\forall_x [x \in P \Longrightarrow x + 1 \in P]$'. [There would be similar changes in steps (1), (2), and (9) to make them conform to the form of (*) in question.]

In conformity with our usual cavalier treatment of 'and', step (6) in the column proof given above might be omitted. If so, the restriction '[$a \in P$]' would be entered at step (7), and the comment for step (7) would be '[(3), (5)]'. [In the modification of the proof suggested to conform with the use of (iii) in place of (*), step (6) could, also, be omitted. If so, the comment for step (7) would be '[(3), (5), (5.5)]'.]

*

As indicated above, (*) is an abbreviation for:

(iii) $\quad \forall_S [(1 \in S \text{ and } \forall_x [x \in S \Longrightarrow x + 1 \in S]) \Longrightarrow I^+ \subseteq S]$

A set S such that

$$\forall_n [n \in S \Longrightarrow n + 1 \in S]$$

may, conveniently, be called <u>an</u> <u>inductive</u> <u>set</u> [cf. 'hereditary property' in the COMMENTARY for page 7-49].

There is a point, illustrated in the Solution for the Sample, which it may be well to stress before assigning the exercises. Using the notation introduced on TC[7-19]a, we can state a rule of reasoning for handling brace-notation:

Each sentence of the form:

(*) $$s \in \{x: F(x)\} \Longleftrightarrow F(s)$$

is logically valid.

So, in particular [see Solution]:

$$n \in \{m: 3 + m \in I^+\} \Longleftrightarrow 3 + n \in I^+$$

is a logically valid sentence.

Rather than presenting students with (*), it is probably better to ask for simpler ways to write sentences like:

$3 \in \{x: x < 5\}$, $a + 2 \in \{y: y - 2 = 3y\}$, etc.

[Answers: $3 < 5$; $(a + 2) - 2 = 3(a + 2)$]

Answers for Part A.

[First, the statement to be proved, then a proof (or a counter-example).]

1. $\forall_n [n > 1 \Longrightarrow n + 1 > 1]$; Suppose that $q > 1$. Since, by Theorem 90, $q + 1 > q$, it follows, by Theorem 86c, that $q + 1 > 1$. Hence, if $q > 1$ then $q + 1 > 1$. Consequently, $\forall_n [n > 1 \Longrightarrow n + 1 > 1]$.

2. $\forall_n [n - 5 \in I^+ \Longrightarrow (n + 1) - 5 \in I^+]$; Suppose that $q - 5 \in I^+$. It follows, by (I^+_2), that $(q - 5) + 1 \in I^+$. But, $(q - 5) + 1 = (q + 1) - 5$. So, $(q + 1) - 5 \in I^+$. Hence, if $q - 5 \in I^+$ then $(q + 1) - 5 \in I^+$. Consequently, ...

3. $\forall_n [n^2 > n \Longrightarrow (n + 1)^2 > n + 1]$; Suppose that $q^2 > q$. It follows, by Theorem 86d, that $q^2 + (2q + 1) > q + (2q + 1)$. By Theorem 101 [and Theorem 83], $q > 0$. Since $2 > 0$, it follows by Theorem 86e that $2q > 2 \cdot 0 = 0$. So, by Theorem 86d, $2q + (q + 1) > 0 + (q + 1) = q + 1$. Since $2q + (q + 1) = q + (2q + 1)$, it follows, by Theorem 86c, that $q^2 + (2q + 1) > q + 1$--that is, $(q + 1)^2 > q + 1$. Hence, if $q^2 > q$ then $(q + 1)^2 > q + 1$. Consequently, ...

4. $\forall_n [n < 4 \Longrightarrow n + 1 < 4]$; 3 is a counter-example.

☆5. $\forall_n [n < 0 \Longrightarrow n + 1 < 0]$; By Theorem 101 [and Theorem 83], for any positive integer q, $q > 0$. So, by Theorem 86b, $q \not< 0$. In particular, if $q + 1 \not< 0$ then $q \not< 0$. Hence [by contraposition], if $q < 0$ then $q + 1 < 0$. Consequently, ...

[See Review Exercise 23 on pages 7-138 and 7-139.]

*

Answer for Part B.

By (I^+_1), $1 \in I^+$. So, by (I^+_2), $1 + 1 \in I^+$ and, again by (I^+_2), $(1 + 1) + 1 \in I^+$. But, by definition, $(1 + 1) + 1 = 3$. Hence, $3 \in I^+$.

Correction. On page 7-51, line 11 should begin:

[Notice that, since $3 \in I^+$, this ---

The explanation called for in connection with (ii) near the bottom of page 7-51 is that (ii) is equivalent to:

$$\forall_n [n \in \{x: 3 + x \in I^+\} \Longrightarrow n + 1 \in \{x: 3 + x \in I^+\}]$$

This point has already been taken up in connection with Part A on page 7-50 [see COMMENTARY for page 7-50].

*

Step (2) of part (i) on page 7-52 is required because the domain of 'n' in step (1) is I^+. Similarly, step (4) of part (ii) is needed to guarantee that (6) is an instance of (5).

That (12) and (13) follow from (3) and (11), respectively, and (17) follows from (16) is explained as in the first paragraph of this COMMENTARY.

*

Notice steps (4) and (17) in the proof on page 7-52. The experience your students have had with quantifiers [and with conditionalizing] will obviate the commonly-found objection of students to induction proofs: You're assuming what you're trying to prove!

(9)	$3q + 3 = 3 + 3q$	[(8)]
(10)	$\forall_n \; 3 + n \in I^+$	[theorem]
(11)	$3 + 3q \in I^+$	[(5), (10)]
(12)	$3q + 3 \in I^+$	[(9), (11)]
(13)	$3(q + 1) \in I^+$	[(7), (12)]
(14)	$3q \in I^+ \Longrightarrow 3(q + 1) \in I^+$	[(13); *(5)]
(15)	$\forall_n \; [3n \in I^+ \Longrightarrow 3(n + 1) \in I^+]$	[(5) - (14)]

<u>Part (iii)</u>:

(16)	$\forall_n \; 3n \in I^+$	[(4), (15), PMI]

2. (i) Since $3 \cdot 1 = 3$, and since, by an earlier theorem, $3 \in I^+$, it follows that $3 \cdot 1 \in I^+$.

   (ii) Suppose that $3q \in I^+$. Since $3(q + 1) = 3 + 3q$, it follows, by an earlier theorem, that $3(q + 1) \in I^+$. Hence, $\forall_n \; [3n \in I^+ \Longrightarrow 3(n + 1) \in I^+]$.

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n \; 3n \in I^+$.

2. (10) $1 \in \{x: x \in P\}$ [(1)]

(11) $\forall_n\ [n \in \{x: x \in P\} \Longrightarrow n + 1 \in \{x: x \in P\}]$ [(9)]

(12) $\forall_S\ [(1 \in S \text{ and } \forall_n\ [n \in S \Longrightarrow n + 1 \in S]) \Longrightarrow \forall_n\ n \in S]$ [(I^+_3)]

(13) $\left\{ \begin{array}{l} (1 \in \{x: x \in P\} \text{ and} \\ \forall_n\ [n \in \{x: x \in P\} \Longrightarrow n + 1 \in \{x: x \in P\}]) \\ \Longrightarrow \forall_n\ n \in \{x: x \in P\} \end{array} \right\}$ [(12)]

(14) $\forall_n\ n \in \{x: x \in P\}$ [(10), (11), (13)]

(15) $\forall_n\ n \in P$ [(14)]

*

Answers for Part B.

1. <u>Part (i)</u>:

(1) $\forall_x\ x \circ 1 = x$ [pm1]

(2) $3 \cdot 1 = 3$ [(1)]

(3) $3 \in I^+$ [theorem]

(4) $3 \cdot 1 \in I^+$ [(2), (3)]

<u>Part (ii)</u>:

(5) $3q \in I^+$ [inductive hypothesis]*

(6) $\forall_x \forall_y\ x(y + 1) = xy + x$ [theorem]

(7) $3(q + 1) = 3q + 3$ [(6)]

(8) $\forall_x \forall_y\ x + y = y + x$ [cpa]

Answers for Part A.

1. Part (i):

(1)	$1 \in P$	[Theorem 82]

Part (ii):

(2)	$q \in P$	[inductive hypothesis]*
(3)	$\forall_x\ x + 1 > x$	[Theorem 90]
(4)	$q + 1 > q$	[(3)]
(5)	$\forall_x \forall_y \forall_z\ [(x > y \text{ and } y > z) \Longrightarrow x > z]$	[Theorem 86c]
(6)	$(q + 1 > q \text{ and } q > 0) \Longrightarrow q + 1 > 0$	[(5)]
(7)	$q + 1 \in P$	[(2), (4), (6)]
(8)	$q \in P \Longrightarrow q + 1 \in P$	[(7); *(2)]
(9)	$\forall_n\ [n \in P \Longrightarrow n + 1 \in P]$	[(2) - (8)]

Part (iii):

(10)	$\forall_n\ n \in P$	[(1), (9), PMI]

[In deriving step (7), Theorem 83 has been used to translate (2) into '$q > 0$' and the consequent '$q + 1 > 0$' of (6) into '$q + 1 \in P$'. As has been pointed out earlier, there is no harm in making such implicit use of Theorem 83. Doing so is analogous, for example, to replacing '$0 > a$' by '$a < 0$' without explicit reference to a definition of '$<$'.]

A proof for Theorem 102, following the procedure outlined on page 7-57, is given on TC[7-58, 59]a. Note that the first ten steps constitute an inductive test-pattern for instances of the generalization '$\forall_m \forall_n\, m + n \in I^+$' --that is, they constitute a test-pattern for sentences obtained by replacing the 'p' in:

$$\forall_n\, p + n \in I^+$$

by a numeral for a positive integer. So, Theorem 102 [step (11)] follows from (1) - (10) by the test-pattern principle.

Part (iii) of the proof in question can be expanded in the form given on page 7-53--merely replace, there, '$F(x)$' [8 times] by '$p + x \in I^+$', and replace '$F(n)$' [once] by '$p + n \in I^+$'. In other words, the instance of (I^+_3) under consideration is that for which

$$S = \{x\colon p + x \in I^+\}.$$

Note that since steps (1) - (10) constitute a <u>test-pattern</u>, it is to be expected that '$\{x\colon p + x \in I^+\}$' is not the name of a set--it becomes so when 'p' is replaced by a numeral.

*

In connection with the text on page 7-58 which is interpolated between Part A and Part B of the exercises, notice that, by the rule (R_1) [see TC[7-36]d], the premiss '$p \in I^+$', which is needed as an auxiliary premiss [see top of TC[7-36]f], might be introduced as a restriction on step (3), rather than as step (2). To see what would happen, think of '$p \in I^+$' having been introduced as a restriction on step (2) of the proof on TC[7-58, 59]a. The resulting restricted test-pattern would justify, as step (10), the conclusion:

$$\forall_{n \in I^+}\, p + n \in I^+$$

But, since the domain of 'n' is I^+, this is redundant. Neither the restriction nor the restricted quantifier is necessary.

Answers for Part A.

1. Part (i):

(1)	$\forall_n\ n + 1 \in I^+$	$[(I^+_2)]$
(2)	$p + 1 \in I^+$	$[(1)]$

Part (ii):

(3)	$p + q \in I^+$	[inductive hypothesis]*
(4)	$(p + q) + 1 \in I^+$	[(1), (3)]
(5)	$\forall_x \forall_y\ (x + y) + 1 = x + (y + 1)$	[theorem]
(6)	$(p + q) + 1 = p + (q + 1)$	[(5)]
(7)	$p + (q + 1) \in I^+$	[(4), (6)]
(8)	$p + q \in I^+ \Longrightarrow p + (q + 1) \in I^+$	[(7); *(3)]
(9)	$\forall_n\ [p + n \in I^+ \Longrightarrow p + (n + 1) \in I^+]$	[(1), (3) - (8)]

Part (iii):

(10)	$\forall_n\ p + n \in I^+$	[(2), (9), PMI]
(11)	$\forall_m \forall_n\ m + n \in I^+$	[(1) - (10)]

[The bar between '(10)' and '(11)' is to indicate that (1) - (10) constitute the inductive proof. Step (11) does not belong to Part (iii). For a discussion of Part (i) of the proof given above, see text on page 7-58.]

2. (i) By (I^+_2), $p + 1 \in I^+$.

   (ii) Suppose that $p + q \in I^+$. It follows, by (I^+_2), that $(p + q) + 1 \in I^+$. Since $(p + q) + 1 = p + (q + 1)$, $p + (q + 1) \in I^+$. Hence, $\forall_n\ [p + n \in I^+ \Longrightarrow p + (n + 1) \in I^+]$.

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\ p + n \in I^+$. Consequently, $\forall_m \forall_n\ m + n \in I^+$.

*

Answers for Part B.

1. Step (14) is, obviously, '$\forall_n$ pn ϵ I^+', the comment being '[(5), (13), PMI]' Since (1) - (14) must be a test-pattern for (15), the comment for (15) is '[(1) - (14)]'.

   To complete part (i), step (3) must be '$\forall_x$ x1 = x' and step (4) is the instance 'p1 = p', the comment being '[(3)]'.

   To complete part (ii), (6) - (12) is a test-pattern for (13), so, the comment for (13) is '[(6) - (12)]', and (12) is 'pq ϵ $I^+ \Longrightarrow$ p(q + 1) ϵ I^+'. From the comment for (12), it is clear that (11) is 'p(q + 1) ϵ I^+'. Step (11) would follow from (10) and 'p(q + 1) = pq + p'. This latter is an instance of (7). So, (8) is 'p(q + 1) = pq + p', the comment for (8) is '[(7)]', and the comment for (11) is '[(8), (10)]'. Finally, by (6), (10) is an instance of Theorem 102. So, (9) is '$\forall_m \forall_n$ m + n ϵ I^+', the comment for (9) is 'Theorem 102', and the comment for (10) is '[(6), (9)]'.

*

Analyzing Exercise 1 in this way may help students in constructing inductive proofs. After applying the suggestion on page 7-55 to write down the conclusions of parts (i) and (ii), one asks oneself questions of the form "Now I could prove this if I could prove that, but, how can I prove that?". More specifically:

> I could get (13) from 'pq ϵ $I^+ \Longrightarrow$ p(q + 1) ϵ I^+' and I could get this from 'p(q + 1) ϵ I^+' and, at the same time, discharge an assumption 'pq ϵ I^+'. But, how can I get 'p(q + 1) ϵ I^+'? Well, I know that p(q + 1) = pq + p and, by Theorem 102, I know that, assuming that pq ϵ I^+, pq + p ϵ I^+. So [by the substitution principle for equations], it follows that p(q + 1) ϵ I^+.

*

2. (i) Since p ϵ I^+ and p1 = p, p1 ϵ I^+

   (ii) Suppose that pq ϵ I^+. By Theorem 102, it follows that pq + p ϵ I^+. But, pq + p = p(q + 1). So, p(q + 1) ϵ I^+. Hence, $\forall_n$ [pn ϵ $I^+ \Longrightarrow$ p(n + 1) ϵ I^+].

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n$ pn ϵ I^+. Consequently, $\forall_m \forall_n$ mn ϵ I^+.

x is even, and, of course, if z is odd then $z - 1$ is even. So, in either case, $x(z - 1)$ is even, and, for $z \in I$, $x(z - 1)/2 \in I$. So, another set of solutions is $\{(x, y, z): x = z - 2, y = (z - 1)(z - 2)/2,$ and $z \in I\}$. Assuming that $z - x = 3$, it follows from the given equation that $y = x(z - 1)/3$. In this case, $y \in I$ only for certain $z \in I$. For example, if $z = 5$ then $x = 2$ and $x(z - 1)/3 = 8/3 \notin I$. The set of solutions in this case is $\{(x, y, z): x = z - 3,$ $y = (z - 3)(z - 1)/3$, $z \in I$, and either z or $z - 1$ is a multiple of $3\}$. For examples: $3 \times (5 + 6) = 3 + (5 \times 6)$, and $4 \times (8 + 7) = 4 + (8 \times 7)$

The complete solution set is, of course, the union of all the sets of solutions one could obtain in the above manner.

There are other ways of obtaining sets of solutions. For example, the given equation is equivalent to '$x(y + z - 1) = yz$' and its solutions can be classified according to the values of '$y + z - 1$'.

Consequently, for any points P, Q, R, and S, either JKLM is a parallelogram or J, K, L, and M are collinear.

Finally, here is a solution of Exercise 9 by analytic geometry:

$$A(1, 0), \quad B(1+a, b), \quad C(a, b), \quad \text{and} \quad D(0, 0).$$

Then, if the coordinates of P_1 are (x, y), it follows from the midpoint formula that

P_2 has coordinates $(2 - x, -y)$,

P_3, $(2 + 2a - 2 + x, 2b + y)$--that is, $(2a + x, 2b + y)$,

P_4, $(-x, -y)$,

and P_5, (x, y).

So, $P_5 = P_1$.

*

10. (a) $\dfrac{3p + 5}{2p - 3}$; $\dfrac{3p - 1}{2p - 7}$ (b) $(p + 1)p$; $(p - 1)(p - 2)$

(c) $(n + 1)(n + 2)(n + 3)$; $(n - 1)n(n + 1)$

(d) $(p + 1)^2 - p^2$; $(p - 1)^2 - (p - 2)^2$

(e) $(n + 1)^2$; $(2n + 1)^2$

11. We want solutions of '$y(z - x) = x(z - 1)$'. It is easy to find sets of solutions (x, y, z) for assumed values of '$z - x$'. For example, assuming that $z - x = 0$, it follows from the given equation that either $x = 0$ or $z = 1$. So, one set of solutions is $\{(x, y, z)\colon y \in I$ and $x = 0 = z\}$ and another is $\{(x, y, z)\colon y \in I$ and $x = 1 = z\}$. Assuming that $z - x = 1$, it follows that $y = x(z - 1)$. So, another set of solutions is $\{(x, y, z)\colon x = z - 1,\ y = (z - 1)^2, \text{ and } z \in I\}$. Assuming that $z - x = -1$, we obtain the set of solutions $\{(x, y, z)\colon x = z + 1,\ y = z^2 - 1, \text{ and } z \in I\}$. Assuming that $z - x = 2$ [the case illustrated in the exercise], it follows that $y = x(z - 1)/2$. Since $x = z - 2$, it follows that if z is even then

*

The more general theorem used in the preceding solution may be accepted as an "obvious" generalization of the theorem concerning the midpoints of the sides of a quadrilateral. However, for completeness, here is a proof of the more general theorem:

If P, Q, R, and S are collinear then, obviously, J, K, L, and M are collinear.

If P, Q, R, and S are noncollinear then some three of these points are noncollinear. We need treat only the case in which, say, P, Q, and R are noncollinear, the cases in which Q, R, and S, or R, S, and P, or S, P, and Q are noncollinear being similar. Suppose, then, that P, Q, and R are noncollinear. It follows that if both Q, R, and S, and S, P, and Q are collinear, then S = Q. In this case, J = M and K = L and, so, J, K, L, and M are collinear. Otherwise, either Q, R, and S, or S, P, and Q are noncollinear. We need consider only the case in which, say, S, P, and Q are noncollinear.

Now, since P, Q, and R are noncollinear, it follows by a well-known theorem [Theorem 6-24 on page 6-128 of Unit 6] that $\overline{JK} \parallel \overline{PR}$. Similarly, if R, S, and P are noncollinear then $\overline{LM} \parallel \overline{PR}$; while if R, S, and P are collinear then $\overline{LM} \subseteq \overleftrightarrow{PR}$. In either case, either $\overline{JK} \parallel \overline{LM}$ or J, K, L, and M are collinear [the last occurs if $\overleftrightarrow{QS} \parallel \overleftrightarrow{PR}$].

Since S, P, and Q are noncollinear, it follows in a similar manner that either $\overline{KL} \parallel \overline{MJ}$ or K, L, M, and J are collinear.

Hence, if P, Q, R, and S are noncollinear then either JKLM is a parallelogram or J, K, L, and M are collinear.

Correction. On page 7-60, line 8b should end:

--- and $\forall_n\ g(n - 1) =$.

and line 7b should end:

--- and $\forall_p\ q(p - 1) =$.

Answers for Miscellaneous Exercises.

[easy: 2, 3, 6, 7, 8, 10; medium: 1, 4, 5; hard: 9, 11]

1. 8 yards; 6 yards

2. 56 years; 32 years

3. 8 days

4. 30 square feet

5. $x^2 + y^2 = 25$ and $y \neq 0$

6. 1120 [Here is a rapid procedure for estimating the velocity:

$$\frac{5 \times 10^4 \times 3600}{2.54 \times 12 \times 5280} = \frac{15 \times 10^6}{2.54 \times 5280} = \frac{6 \times 10^6}{5280} = \frac{12 \times 10^6}{10^4} = 1200$$

Since $2.54 > 5/2$ and $5280 > 10^4/2$, the velocity is somewhat less than 1200 miles per hour.]

7. 7.3×10^7

8. (a) quartered; (b) doubled; (c) quadrupled

9. [As is well-known, the midpoints of the sides of any quadrilateral are vertices of a parallelogram (see Exercise 9 on page 6-169 of Unit 6). It is not difficult to prove that, more generally, if P, Q, R, and S are any points, not necessarily different, and J, K, L, and M are the midpoints of $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$, respectively, then either JKLM is a parallelogram or J, K, L, and M are collinear. Using this theorem it is easy to solve the present exercise.]

Let E be the midpoint of $\overline{P_4P_1}$. Since ABCD is a parallelogram, it follows that A, B, and C are noncollinear. Hence, A, B, C, and E are noncollinear and [by the theorem stated above] ABCE is a parallelogram. Consequently, D = E. Since D is the midpoint of $\overline{P_4P_5}$ and E is the midpoint of $\overline{P_4P_1}$, it follows that $P_5 = P_1$.

introduces a defined term--'perpendicular'--but, at the same time, makes it dispensable.]

On the other hand, while the recursive definition given for the function T makes it possible to calculate, by recursion, the value of T for any numerical argument, this recursive definition does not furnish a procedure for replacing an expression such as 'T_{2p+3}' by an equivalent expression which does not contain 'T'. One may replace 'T_{2p+3}' by '$T_{2p+2} + (2p + 3)$'--and, recursively, by other expressions--but all such replacements contain 'T'.

In view of this difference between explicit definitions and recursive definitions, one may well ask what justification one has in assuming that a recursive definition does actually define a function. Is there a function whose domain is I^+ and which satisfies the given recursive definition? Is there more than one such function? Unless the answer to the first question is 'yes' and the answer to the second question is 'no', the recursive definition is not worthy of being considered a definition.

In the case of T [and of many other functions which will be defined recursively] the desired answers to the two questions can be justified by finding an explicit definition [(2) on page 7-63] for the triangular numbers. One can then justify the affirmative answer to the first of the two questions by showing that the right side of the explicit definition satisfies the recursive definition [see lines 4-9 on page 7-63]. Using this fact one can justify the negative answer to the second question by proving that the only function which satisfies the recursive definition is the one given by the explicit definition [see page 7-64, and the related COMMENTARY].

In the case of the recursive definitions for f and r on page 7-62, there are no corresponding explicit definitions. In such cases we shall assume without question that each recursive definition does characterize a unique function. This assumption is, in fact, justified. However, the proof that each recursive definition is satisfied by at least one function is somewhat beyond the level of this COMMENTARY [and far beyond the level of the text]. The proof that each recursive definition is satisfied by at most one function is given in the COMMENTARY for pages 7-63 and 7-64. For an excellent exposition of these matters, see the paper "On Mathematical Induction", by Leon Henkin, in The American Mathematical Monthly for April 1960--vol. 67, pages 323-338.

$f_1 = 1$; $f_2 = 2$; $f_3 = 6$; $f_4 = 24$ [f is the factorial function (see Unit 8)]

$r_1 = 1$; $r_2 = \frac{3}{2}$; $r_3 = \frac{17}{12}$; $r_4 = \frac{577}{408}$

[The values of r are approximations to $\sqrt{2}$ (see Unit 3). Students should recognize the second equation of the definition as describing the divide-and-average procedure for finding approximations to $\sqrt{2}$. What equation describes the procedure for finding approximations to $\sqrt{3}$?]

Exercise 10 on page 7-60 should have prepared students for the fill-in in the middle of page 7-62:

$$\forall_n \; f_{n+2} = (n + 2)f_{n+1} = (n + 2)(n + 1)f_n$$

*

The extent to which a recursive definition, such as:

$$\begin{cases} T_1 = 1 \\ \forall_n \; T_{n+1} = T_n + (n + 1) \end{cases}$$

differs from an explicit definition, such as:

$$\forall_n \; S_n = n^2$$

can be appreciated by the following consideration.

The explicit definition given for the function S not only furnishes a procedure for calculating the value of S for a given argument, but also furnishes a means for eliminating an expression of the form 'S...' from any context. For example, one may replace 'S_{2p+3}' by the expression '$(2p + 3)^2$', in which 'S' does not occur. Evidently, the explicit definition introduces 'S' as a sort of shorthand which one could dispense with. [The introduction of 'S' by explicit definition is entirely analogous to the introduction of the word 'perpendicular' in geometry. In Unit 6, an explicit definition tells us that a sentence of the form '... and ___ are perpendicular lines' is just a short way of saying 'the union of the lines ... and ___ contains a right angle'. Here again, the explicit definition

In dealing with the triangular numbers, the Greeks thought of what we now call the [nonzero, finite] cardinal numbers--the "counting numbers". It fits in better with our discussion to define the triangular numbers to be the corresponding positive integers.

*

$T_6 = 21$, $T_7 = 28$

Some students should notice that, for example, $T_6 = T_5 + 6$, and that $T_7 = T_6 + 7$, and, so, be able to guess that $T_{11} = T_{10} + 11$. Hence, on being told that $T_{10} = 55$, they can guess that $T_{11} = 66$. Also, given that $T_{25} = 325$, $T_{26} = 351$.

For your information, (2) on page 7-63 gives a quick way to compute any triangular number. You can use this [until students discover it] if you need a source of additional examples like those just discussed.

*

The generalization '$\forall_n \ T_{n+1} = T_n + (n + 1)$' formulates what students should have seen when answering the questions in the text. From the figures at the top of the page one sees that, for each n, the nth triangle has n dots on a side, and that each triangle is obtained from the preceding one by adjoining to it a slanting line of dots which contains one more dot than the number in a side of the preceding triangle.

*

Computing T_8 by recursion:

$$\begin{aligned} T_8 &= T_7 + 8 \\ &= T_6 + 7 + 8 \\ &= 21 + 7 + 8 = 36 \end{aligned}$$

*

(8, 36) and (9, 45) belong to T.

*

The only difference between the derivation given above and that on page 7-64 is the use of 'g' instead of 'T' and of the assumptions concerning h in place of the theorems in steps (2) and (8) of page 7-64.

Notice that in the derivation above, the precise form of the right sides of the four assumptions played no real part. For example, in place of:

$$\forall_n \; g_{n+1} = g_n + (n + 1) \qquad \text{and:} \qquad \forall_n \; h_{n+1} = h_n + (n + 1)$$

one might as well have had:

$$\forall_n \; g_{n+1} = g_n \cdot (n + 1) \qquad \text{and:} \qquad \forall_n \; h_{n+1} = h_n \cdot (n + 1)$$

The same argument, with these replacements, shows that at most one function satisfies the recursive definition given for f on page 7-62.

More generally, one says that a function is defined recursively in terms of a number, a, and a function, F, of two variables, if it satisfies the recursive definition:

$$g_1 = a$$
$$\forall_n \; g_{n+1} = F(g_n, n)$$

[For the recursive definition (1) on page 7-61, $a = 1$ and $F(x, y) = x + (y + 1)$; for the recursive definition of f on page 7-62, $a = 1$ and $F(x, y) = (y + 1) \cdot x$; for the recursive definition of r on page 7-62, $a = 1$ and $F(x, y) = (x + \frac{2}{x})/2$.] Only minor changes in steps (1), (2), and (5) - (9) of the derivation above suffice to change it into a test-pattern for the generalization:

> For each number a, and each function F, at most one function is defined recursively in terms of a and F.

As remarked at the end of the COMMENTARY for page 7-62, the proof of the complementary generalization obtained by replacing 'most' by 'least' goes beyond the limits of this COMMENTARY.

*

The role of the inductive proof in establishing that there is at most one function which satisfies the recursive definition (1) is clarified by the considerations which follow.

Suppose that g and h satisfy the recursive definition (1)--that is, assume that

$$\begin{cases} g_1 = 1 \\ \forall_n\ g_{n+1} = g_n + (n+1) \end{cases} \quad \text{and that} \quad \begin{cases} h_1 = 1 \\ \forall_n\ h_{n+1} = h_n + (n+1) \end{cases}$$

From these assumptions one can show, just as on page 7-64, that $\forall_n\ g_n = h_n$--that is, that $g = h$. Let's do it.

Part (i):

(1)	$g_1 = 1$	[assumption]
(2)	$h_1 = 1$	[assumption]
(3)	$g_1 = h_1$	[(1), (2)]

Part (ii):

(4)	$g_q = h_q$	[inductive hypothesis]*
(5)	$\forall_n\ g_{n+1} = g_n + (n+1)$	[assumption]
(6)	$g_{q+1} = g_q + (q+1)$	[(5)]
(7)	$g_{q+1} = h_q + (q+1)$	[(4), (6)]
(8)	$\forall_n\ h_{n+1} = h_n + (n+1)$	[assumption]
(9)	$h_{q+1} = h_q + (q+1)$	[(8)]
(10)	$g_{q+1} = h_{q+1}$	[(7), (9)]
(11)	$g_q = h_q \Rightarrow g_{q+1} = h_{q+1}$	[(10); *(4)]
(12)	$\forall_n\ [g_n = h_n \Rightarrow g_{n+1} = h_{n+1}]$	[(4) - (11)]

Part (iii):

(13)	$\forall_n\ g_n = h_n$	[(3), (12), PMI]

The checking referred to in the first paragraph on page 7-63 shows that the function T which is [explicitly] defined by (2) does satisfy the recursive definition (1) given on page 7-61.

The inductive proof discussed in the third paragraph on page 7-63 [and given on page 7-64] shows that if T is <u>any</u> function which satisfies the recursive definition [premisses (1) and (5) on page 7-64] then T is the particular function which is explicitly defined by (2) [conclusion (13) on page 7-64]. The proof makes use of the fact, previously established, that the function defined by (2) satisfies the recursive definition [steps (2) and (8) on page 7-64].

[The use of 'prove' in the first line of the third paragraph on page 7-63 is slightly improper. More properly, our job is to derive (2) from (1).]

The two jobs, of establishing the existence and the uniqueness of a function T which satisfies (1), are often carried out simultaneously by incorporating, in parts (i) and (ii), respectively, of the inductive proof, the proofs of the theorems in steps (2) and (8). This is the actual effect of the suggestion at the top of page 7-65. It is up to you to decide how much detail your students are required to put in in such "algebra steps". You may even wish to require more detail of weaker students than of stronger ones.

The inference indicated at the bottom of the tree-diagram on page 7-65 is, really, a combination of five inferences, as in the form for part (iii) of an inductive proof given on page 7-53. In detail:

[There is a sixth inference:

$$\frac{(3') \qquad (12')}{(3') \text{ and } (12')}$$

not allowed for in the form on page 7-53.]

$$\dfrac{\dfrac{\begin{matrix}\vdots\\(3)\end{matrix}}{(3')} \qquad \dfrac{\begin{matrix}\vdots\\(12)\end{matrix}}{(12')} \qquad \dfrac{[(I^+_3)]}{[\text{instance}]}}{\dfrac{(13')}{(13)}}$$

Here, (3′) is '$1 \in \{m\colon T_m = \frac{m(m+1)}{2}\}$',

(12′) is '$\forall_n\, [n \in \{m\colon T_m = \frac{m(m+1)}{2}\} \Rightarrow n + 1 \in \{m\colon T_m = \frac{m(m+1)}{2}\}]$',

(13′) is '$\forall_n\, n \in \{m\colon T_m = \frac{m(m+1)}{2}\}$' and '[instance]' refers to an instance of (I^+_3) which may be described as '((3′) and (12′)) $\Longrightarrow$ (13′)'.

Answers for Part A.

1. (a) 4; 6; 8; 10 (b) $2n$

(c) [The column proof on page 7-64 can serve as a model for a column proof for Exercise 1(c). The paragraph proof given below should answer any questions as to how to change page 7-64 into an answer for Exercise 1(c).]

(i) Since, by the recursive definition of E, $E_1 = 2$, and since $2 \cdot 1 = 2$, $E_1 = 2 \cdot 1$.

(ii) Suppose that $E_q = 2q$. Since, by the recursive definition, $E_{q+1} = E_q + 2$, it follows that $E_{q+1} = 2q + 2$. But, $2q + 2 = 2(q + 1)$. So, if $E_q = 2q$ then $E_{q+1} = 2(q + 1)$. Consequently, $\forall_n\ [E_n = 2n \Longrightarrow E_{n+1} = 2(n + 1)]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\ E_n = 2n$.

2. $$\begin{cases} O_1 = 1 \\ \forall_n\ O_{n+1} = O_n + 2 \end{cases}$$

(a) 3; 5; 7; 9 (b) $2n - 1$ (c) [Use Exercise 1 as a model.]

3. (a) 4; 9; 16; 25 (b) n^2 (c) [Use Exercise 1 as a model.]

$S_{2(q+1)+1} = 8T_{q+1} + 1$. Consequently, $\forall_n [S_{2n+1} = 8T_n + 1 \Longrightarrow S_{2(n+1)+1} = 8T_{n+1} + 1]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\ S_{2n+1} = 8T_n + 1$.

As in Exercise 1(c) of Part A, the proof, above, depends on the fact that the functions g and h such that, for each n, $g_n = S_{2n+1}$ and $h_n = 8T_n + 1$ satisfy the same recursive definition.

The theorem of Exercise 2 gives a simple test for telling whether a positive integer N is a triangular number. It is if (and only if) $8N + 1$ is a square.

3. $\forall_n\ S_n + n = 2T_n$

(i) $S_1 + 1 = 1 + 1 = 2 \cdot 1 = 2T_1$

(ii) Suppose that $S_q + q = 2T_q$. It follows from this and the r.d.s of S and T that $S_{q+1} + (q + 1) = S_q + (2q+1) + (q+1) = (S_q + q) + 2(q+1) = 2T_q + 2(q+1) = 2T_{q+1}$. So, if $S_q + q = 2T_q$ then $S_{q+1} + (q + 1) = 2T_{q+1}$. Consequently, $\forall_n [S_n + n = 2T_n \Longrightarrow S_{n+1} + (n+1) = 2T_{n+1}]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\ S_n + n = 2T_n$.

It is, of course, permissible to state the generalization in a different form, for example, as '$\forall_n\ T_n = \frac{n + S_n}{2}$' or as '$\forall_n\ 2T_n - S_n = n$', and to give an appropriate inductive proof.

4. [The theorem to be proved is: $\forall_n\ E_n = \frac{O_1 + O_{2n}}{2}$
Either an inductive proof, based on the r.d.s of E and O (see Part A) or a direct proof, based on the explicit definitions, is acceptable.]

and T already obtained: Since $S_{q+1} = (q + 1)^2$ and $T_q + T_{q+1} =$ $\frac{q(q + 1)}{2} + \frac{(q + 1)(q + 2)}{2} = \frac{(q + 1)(q + q + 2)}{2} = (q + 1)^2$, it follows that $S_{q+1} = T_q + T_{q+1}$. Consequently, $\forall_n\, S_{n+1} = T_n + T_{n+1}$.

Students may suggest the more inclusive generalization '$\forall_n\, S_n = T_{n-1} + T_n$'. This is fine if they also realize that it requires extending the definition of T to include '$T_0 = 0$', and that since they are completely free to define 'T_0' in any way they wish (or to leave it undefined), they may define it in this way if they wish to justify the more inclusive generalization. They may then notice that the so-extended function T can be defined recursively by:

$$\begin{cases} T_0 = 0 \\ \forall_n\, T_n = T_{n-1} + n \end{cases}$$

2. (a) $(2n + 1)^2$

(b) $S_{2q+1} = (2q + 1)^2 = 4q^2 + 4q + 1 = 4q(q + 1) + 1 = 8\frac{q(q + 1)}{2} + 1 =$ $8T_q + 1$. So, $\forall_n\, S_{2n+1} = 8T_n + 1$.

(c) (i) By the r.d.s of S and T, $S_{2\cdot1+1} = S_2 + 5 = (S_1 + 3) + 5 = 9$, and $8T_1 + 1 = 8 \cdot 1 + 1 = 9$. Hence, $S_{2\cdot1+1} = 8T_1 + 1$.

(ii) Suppose that $S_{2q+1} = 8T_q + 1$. Since, by the r.d. of S, $S_{2(q+1)+1} = S_{2q+2} + [2 \cdot 2(q + 1) + 1] = S_{2q+1} + [2(2q + 1) + 1]$ $+ [2 \cdot 2(q + 1) + 1] = S_{2q+1} + 8(q + 1)$, it follows that $S_{2(q+1)+1} = 8T_q + 1 + 8(q + 1) = 8[T_q + (q + 1)] + 1 = 8T_{q+1}$ $+ 1$, by the r.d. of T. So, if $S_{2q+1} = 8T_q + 1$ then

Answers for Part B [which begins on page 7-66].

1. $\forall_n\, S_{n+1} = T_n + T_{n+1}$;

(i) By the recursive definitions of S and T, $S_{1+1} = S_1 + (2 \cdot 1 + 1) = 1 + 3 = 4$, and $T_1 + T_{1+1} = T_1 + [T_1 + (1 + 1)] = 1 + [1 + 2] = 4$. Hence, $S_{1+1} = T_1 + T_{1+1}$.

(ii) Suppose that $S_{q+1} = T_q + T_{q+1}$. Since, by the recursive definition of S, $S_{(q+1)+1} = S_{q+1} + [2(q + 1) + 1]$, it follows that $S_{(q+1)+1} = T_q + T_{q+1} + (2q + 3)$. But, $2q + 3 = (q+1) + (q+2)$. So, $T_q + T_{q+1} + (2q + 3) = [T_q + (q + 1)] + [T_{q+1} + (q + 2)] = T_{q+1} + T_{(q+1)+1}$, by the recursive definition of T. So, if $S_{q+1} = T_q + T_{q+1}$ then $S_{(q+1)+1} = T_{q+1} + T_{(q+1)+1}$. Consequently, $\forall_n\, [S_{n+1} = T_n + T_{n+1} \Longrightarrow S_{(n+1)+1} = T_{n+1} + T_{(n+1)+1}]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\, S_{n+1} = T_n + T_{n+1}$.

Note that the proof just given depends on the fact that if g and h are the functions such that, for each n, $g_n = S_{n+1}$ and $h_n = T_n + T_{n+1}$, then g and h satisfy the same recursive definition [$g_1 = 4 = h_1$, $\forall_n\, g_{n+1} = g_n + (2n + 3)$, $\forall_n\, h_{n+1} = h_n + (2n + 3)$]. Consequently, as shown in the COMMENTARY for page 7-65, g = h--that is, $\forall_n\, S_{n+1} = T_n + T_{n+1}$.

Students may ask why they should use induction to prove the generalization discovered in Exercise 1. The answer is: For practice. As students may argue, it is easier to use the explicit definitions of S

Answers for Part C.

1. 3.333333333

2. 3.333333333

*

If you wish, you can extend some of Exercises 3-7 to exercises like those of Part A on page 7-66, by asking students to guess explicit definitions and, perhaps, to derive these from the given recursive definitions. In case you do, the answers for Exercises 3-7 given below include the appropriate explicit definitions. Although there is no point in telling students this, Exercises 1 and 2 deal with geometric progressions. These are taken up in Unit 8.

*

3. -3 [$\forall_n f_n = -3 + 5(n - 1)$, or: $\forall_n f_n = 5n - 8$]

4. 10 [$\forall_n f_n = 10n - 1$]

5. 14; 8 [$g_1 = 4$ and $\forall_n g_{n+1} = T_n + 4$, or: $\forall_n g_n = T_{n-1} + 4$, if one defines T_0 to be 0. (Some simple connection between g and T is suggested by the '+ n' in the r.d. of g. Noting that $\forall_n g_{(n+1)+1} = g_{n+1} + (n + 1)$, and that $\forall_n T_{n+1} = T_n + (n + 1)$ suggests comparing g_{n+1} and T_n.)]

6. 125 [$\forall_n c_n = n^3$; ($1^3 = 1$, and $\forall_n (n + 1)^3 = n^3 + (3n^2 + 3n + 1)$)]

7. 22 [$\forall_n g_n = S_{n+1} - 3$ (See comment on Exercise 5.)]

8. 1024, or: 2^{10}

9. 256 [Note the abnormal form of the generalization. It is equivalent to '$\forall_{n>3} t_{n-2} = t_{n-3} \cdot 2$' and to '$\forall_n t_{n+1} = t_n \cdot 2$'.]

10. 1/3; 1/100 [much cleverness needed here]

11. 15; 0 [Of course, $\forall_{n > 6}\ f_n = 0.$]

12. 4; There is no m such that $k_m > 10/9$--in fact, $\forall_n\ k_n < 10/9$.

*

Answers for Part D.

1. $(1.03)^4 \cdot 100$, or: 112.550881

2. \$112.55

☆3. $\forall_n\ A_{n-1} = (1.03)^{n-1} \cdot 100$ [Since the domain of 'n' is I^+, the answer '$\forall_n\ A_n = (1.03)^n \cdot 100$' is not quite correct. It fails to specify A_0.]

*

Although students are not asked to check the answer for Exercise ☆3 by deducing it from the recursive definition given for A, some may wish to do so. Such a derivation requires the use of a modification of (I^+_3) which justifies proof by induction over the set $I^+ \cup \{0\}$ of nonnegative integers. One such modification of (I^+_3) is the instance of Theorem 114 [on page 7-99] for j = 0. [In connection with Theorem 114, note that, by a convention introduced on page 7-95, the domain of the variables 'i', 'j', and 'k' is the set of all integers.] This instance will be shown [see TC[7-99]] to be equivalent to:

(*) $\forall_S\ [(0 \in S \text{ and } \forall_n\ [n - 1 \in S \Longrightarrow n \in S]) \Longrightarrow \forall_n\ n - 1 \in S]$,

and (*) lends itself readily to the derivation of:

(1) $\forall_n\ A_{n-1} = (1.03)^{n-1} \cdot 100$

from the recursive definition:

(2) $\begin{cases} A_0 = 100 \\ \forall_n\ A_n = 1.03\ A_{n-1} \end{cases}$

The derivation also requires another recursive definition, that of the exponential sequence with base 1.03:

$$\begin{cases} (1.03)^0 = 1 \\ \forall_n\ (1.03)^n = (1.03)^{n-1} \cdot (1.03) \end{cases}$$

[Such sequences are studied in Unit 8.]

Here is a paragraph derivation, for such as want it, of (1) from (2):

(i) By (2), $A_o = 100 = 1 \cdot 100 = (1.03)^0 \cdot 100 = (1.03)^{1-1} \cdot 100$.

(ii) Suppose that $A_{p-1} = (1.03)^{p-1} \cdot 100$. By (2), $A_p = 1.03\, A_{p-1} = 1.03\, [(1.03)^{p-1} \cdot 100] = [(1.03)^{p-1}\,(1.03)] \cdot 100 = (1.03)^p \cdot 100$.

Consequently, $\forall_n\, [A_{n-1} = (1.03)^{n-1} \cdot 100 \Longrightarrow A_n = (1.03)^n \cdot 100]$.

(ii) From (i) and (ii) it follows, by (*), that

$\forall_n\; A_{n-1} = (1.03)^{n-1} \cdot 100$.

(c) [Proof is very much like that for (a).]

In connection with the r. d. for part (c), note that if g is the function such that, for each n, $g_n = (f_n - 1)/3$ then g and S satisfy the same recursive definition. So, as in the COMMENTARY for page 7-65, $\forall_n (f_n - 1)/3 = S_n$--that is, $\forall_n f_n = 3n^2 + 1$.

*

Although students are not expected to think of them, there are other recursive definitions than those given above for the functions of Exercise 1. For example:

(a) $\begin{cases} f_1 = 6 \\ \forall_n f_{n+1} = f_n \cdot \dfrac{5n+6}{5n+1} \end{cases}$ $\left[\text{from: } \dfrac{f_{n+1}}{f_n} = \dfrac{5(n+1)+1}{5n+1}\right]$

(b) $\begin{cases} f_1 = -2 \\ \forall_n f_{n+1} = -2f_n - 9n \end{cases}$ $[\text{from: } f_{n+1} + 2f_n = (-3n-2) + 2(-3n+1)]$

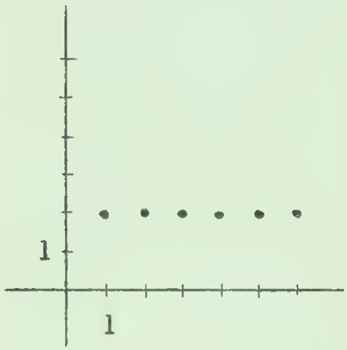
*

2. (i) by the r. d. s for B and T, $B_1 = 2 = 2 \cdot 1 = 2T_1$.

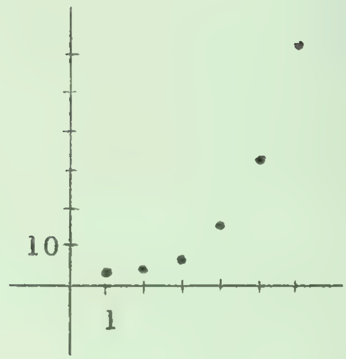
(ii) Suppose that $B_q = 2T_q$. From this and the r. d. s for B and T, it follows that $B_{q+1} = B_q + 2(q+1) = 2T_q + 2(q+1) = 2[T_q + (q+1)] = 2T_{q+1}$. So, if $B_q = 2T_q$ then $B_{q+1} = 2T_{q+1}$. Consequently, $\forall_n [B_n = 2T_n \Rightarrow B_{n+1} = 2T_{n+1}]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n B_n = 2T_n$.

7.



8.



[Compare this function with r on page 7-62.]

*

Answers for Part F.

1. Recursive definitions:

(a) $\begin{cases} f_1 = 6 \\ \forall_n\, f_{n+1} = f_n + 5 \end{cases}$

(b) $\begin{cases} f_1 = -2 \\ \forall_n\, f_{n+1} = f_n - 3 \end{cases}$

(c) $\begin{cases} f_1 = 4 \\ \forall_n\, f_{n+1} = f_n + 3(2n + 1) \end{cases}$

Proofs:

(a) (i) By the r.d., $f_1 = 6 = 5 \cdot 1 + 1$

(ii) Suppose that $f_q = 5q + 1$. Since, by the r.d., $f_{q+1} = f_q + 5$, it follows that $f_{q+1} = (5q + 1) + 5 = 5(q + 1) + 1$. So, if $f_q = 5q + 1$ then $f_{q+1} = 5(q + 1) + 1$. Consequently, $\forall_n\, [f_n = 5n + 1 \Longrightarrow f_{n+1} = 5(n + 1) + 1]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\, f_n = 5n + 1$.

(b) [Proof is very much like that for (a).]

$$\frac{q[mq + (2 - m)]}{2} + (mq + 1) = \frac{mq^2 + (m + 2)q + 2}{2}. \quad \text{But,}$$

$$(q + 1)[m(q + 1) + (2 - m)] = mq^2 + (m + 2)q + 2. \quad \text{Hence, } \ldots .$$

*

There are numerous generalizations involving polygonal numbers. Here are a few:

$$\forall_m \forall_n P^{(m+2)}_{n+1} = (m - 1)T_n + T_{n+1}$$ [See Exercise 1 of Part B on page 7-66.]

$$\forall_m \forall_n P^{(m+2)}_{n+1} = mT_n + (n + 1)$$

$$\forall_m \forall_n P^{(m+3)}_{n+1} = P^{(m+2)}_{n+1} + T_n$$ [See Exercise 2 of Part G on page 7-70.]

$$\forall_m \forall_n P^{(m+2)}_{n} + P^{(m+4)}_{n} = 2P^{(m+3)}_{n}$$ [See Exercise 3 of Part B on page 7-67.]

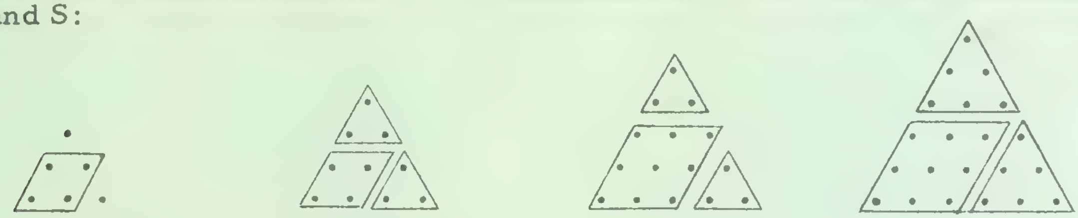
$$\forall_n 3P^{(5)}_n = T_{3n-1}$$

$$\forall_n 3P^{(8)}_n = S_{3n-1} - 1$$

$$\forall_n 2P^{(10)}_n = T_{4n-2} - 1$$

Each of these is easy to derive from the appropriate explicit definitions, and doing so gives practice in simplifying algebraic expressions. None of them is especially difficult to derive inductively from the appropriate recursive definitions.

It is interesting to discover such generalizations by playing around with geometric representations. For example, here is another tie-up between T and S:



$T_3 = S_2 + 2 \cdot T_1$ $\quad$ $T_4 = S_2 + 2 \cdot T_2$ $\quad$ $T_5 = S_3 + 2 \cdot T_2$ $\quad$ $T_6 = S_3 + 2 \cdot T_3$

These four examples suggest two generalizations:

$$\forall_n T_{2n+1} = S_{n+1} + 2T_n \qquad \text{and:} \qquad \forall_n T_{2n} = S_n + 2 \cdot T_n$$

Parts H-L deal with some combinatorial problems. There is more on this subject in Unit 8.

*

Answers for Part H.

1. 6; 10; 0

2. $G_1 = 0$, $\forall_n\, G_{n+1} = G_n + n$ [When a new team joins a league which formerly had n members, n additional games must be played--the old teams play among themselves, as before, and each of them plays one game with the new team.]

3. $\forall_n\, G_n = \frac{n(n-1)}{2}$;

   (i) By the r. d. for G, $G_1 = 0 = \frac{1(1-1)}{2}$.

   (ii) Suppose that $G_q = \frac{q(q-1)}{2}$. From this and the r. d. for G it follows that $G_{q+1} = \frac{q(q-1)}{2} + q = \frac{q(q-1+2)}{2} = \frac{(q+1)[(q+1)-1]}{2}$. So, if $G_q = \frac{q(q-1)}{2}$ then $G_{q+1} = \frac{(q+1)[(q+1)-1]}{2}$. Consequently, $\forall_n\, [G_n = \frac{n(n-1)}{2} \Longrightarrow G_{n+1} = \frac{(n+1)[(n+1)-1]}{2}]$.

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\, G_n = \frac{n(n-1)}{2}$.

*

Notice that $\forall_n\; C(n, 2) = T_{n-1}$ [if $T_0 = 0$]. This could have been foreseen by comparing the r. d. s for G and T [see comment on answer to Exercise 5 of Part C on page 7-68].

Answer for Part I.

$$
\begin{aligned}
C(6, 3) &= C(5, 3) + C(5, 2) \\
&= [C(4, 3) + C(4, 2)] + C(5, 2) \\
&= [C(3, 3) + C(3, 2)] + C(4, 2) + C(5, 2) \\
&= 1 + \frac{3 \cdot 2}{2} + \frac{4 \cdot 3}{2} + \frac{5 \cdot 4}{2} \\
&= 20
\end{aligned}
$$

Answers for Part J.

p \ n	0	1	2	3	4	5	6	7	8	9
0	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0	0	0
3	1	3	3	1	0	0	0	0	0	0
4	1	4	6	4	1	0	0	0	0	0
5	1	5	10	10	5	1	0	0	0	0
6	1	6	15	20	15	6	1	0	0	0
7	1	7	21	35	35	21	7	1	0	0
8	1	8	28	56	70	56	28	8	1	0
9	1	9	36	84	126	126	84	36	9	1

C(5, 0) is the number of 0-membered subsets of a 5-membered set. Since there is just one empty set, $\emptyset$, C(5, 0) = 1.

C(0, 5) is the number of 5-membered subsets of $\emptyset$. So, C(0, 5) = 0.

C(0, 0) = 1--the empty set has just one empty subset, itself.

Having filled in the 0-column with '1's and the remainder of the 0-row with '0's, it is easy to use the recursion formula, first, to complete the entries in the 1-row, next, those in the 2-row, etc.

Students can scarcely avoid making discoveries concerning the numbers C(n, p). For example, it appears from the table that if p + q = n then C(n, p) = C(n, q). Why this is so becomes clear when one realizes that if one pairs each subset of an n-membered set with its complement, it becomes evident that, for each p, an n-membered set has exactly as many p-membered as (n - p)-membered subsets.

This and other such properties of the numbers C(n, p) are studied in Unit 8. One explicit definition for the function C is:

$$\forall_n \forall_p \; C(n, p) = \frac{n(n-1)\ldots(n-p+1)}{1 \cdot 2 \cdot \ldots \cdot p}$$

[Note that, for a given p, the numerator and denominator have p factors each.]

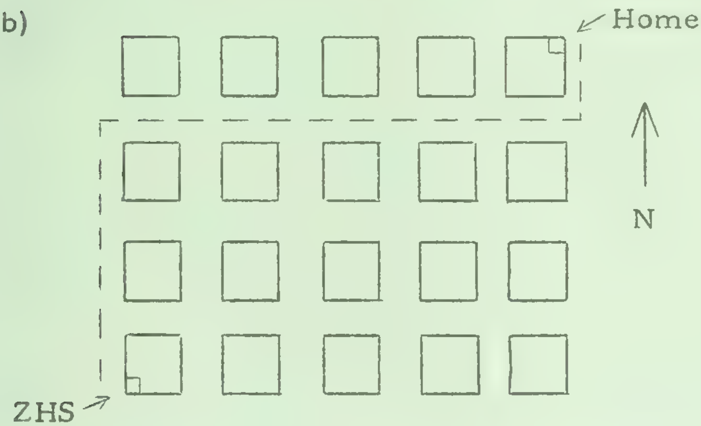
Answers for Part K.

1. Specimen answers:

(a)

1st	2nd	3rd	4th	5th	6th	7th	8th	9th
W	S	W	W	S	W	S	S	W
S	W	W	W	W	W	S	S	S

(b)



2. (a) C(9, 4) [or: 126] [The number of ways in which one can choose 4 of the 9 columns in which to write 'S's is precisely the number of 4-membered subsets of a 9-membered set. From the table in Part J, C(9, 4) = 126. (Of course, one can, instead, concentrate on choosing the 5 columns in which to write 'W's. C(9, 5) is also 126.)] [The possible answer '125', based on the fact that one route has already been described, is a quibble.]

(b) 126 [Each 9-block route can be described by writing 4 'S's and 5 'W's on a line of Milton's notebook, for, at the beginning of each of 9 blocks Milton must decide whether to walk south or west and, to arrive at ZHS, he must choose south 4 times and west 5 times. (If he chooses to walk east or north, Milton will have to walk more than 9 blocks to reach ZHS.) Moreover, each such choice of 4 'S's and 5 'W's describes an actual 9-block route which Milton can take, and two choices describe different routes. So, the number of 9-block routes is exactly the number of ways of choosing 4 of the 9 columns in Milton's notebook.]

☆3. None [assuming that Milton doesn't turn around in the middle of a block].

☆4. 5082 [or: $11 \cdot C(11, 5)$] [On an 11-block route either 4 blocks will be walked south, 1 east, and 6 west, or 5 blocks will be walked south, 1 north, and 5 west. There are several ways of counting these routes: (i) Of Milton's 11 choices (one at the beginning of each block), pick 5. There are $C(11, 5)$ ways of doing this. Then, either pick one of these 5 for an east-choice, or pick one of the remaining 6 for a north-choice. There are, then, $5 \cdot C(11, 5)$ routes of the first kind and $6 \cdot C(11, 5)$ routes of the second kind. So, altogether, there are $11 \cdot C(11, 5)$ 11-block routes. (ii) Another way to count routes of the first kind is to pick one of Milton's 11 choices as an east-choice and 4 of the remaining 10 as south-choices. This gives $11 \cdot C(10, 4)$ such routes. For routes of the second kind, pick 1 of Milton's 11 choices as a north-choice and 5 of the remaining 10 as south-choices. This gives $11 \cdot C(10, 5)$ such routes. Altogether, there are $11[C(10, 5) + C(10, 4)]$ routes. By the recursion formula on page 7-73, this amounts to $11 \cdot C(11, 5)$ routes. (iii) A third way to count routes of the first kind is to pick 4 of Milton's 11 choices as south-choices and 1 of the remaining 7 as an east-choice. This gives $7 \cdot C(11, 4)$ as the number of routes of the first kind. As in (i), there are $6 \cdot C(11, 5)$ routes of the second kind. As expected, $7 \cdot C(11, 4) + 6 \cdot C(11, 5) = 5082$.]

☆5. 1176 [The routes to be counted are those of Exercise 4 in whose descriptions, in Milton's notebook, a 'W' never is adjacent to an 'E' and an 'S' is never adjacent to an 'N'. In the case of a route of the first kind--one whose description contains an 'E', 4 'S's, and 6 'W's--if the 'E' is in the 1st or 11th column, this leaves 9 columns for 'W's, while if the 'E' is in any other column, this leaves 8 columns for 'W's. So, the number of routes of the first kind, as restricted in this exercise, is $2 \cdot C(9, 6) + 9 \cdot C(8, 6)$. Similarly, the number of restricted routes of the second kind--those whose descriptions contain an 'N', 5 'S's, and 5 'W's, with no 'S' adjacent to the 'N'--is $2 \cdot C(9, 5) + 9 \cdot C(8, 5)$. So, altogether, there are $2[C(9, 6) + C(9, 5)] + 9[C(8, 6) + C(8, 5)]$ restricted routes. By the recursion formula on page 7-73, this amounts to $2 \cdot C(10, 6) + 9 \cdot C(9, 6)$ routes.]

Different distributions correspond to different ways of inserting the partitions--that is, to different ways of distinguishing n - 1 among a given row of n - 1 + p objects as being partitions. This is just the number of ways of choosing n - 1 objects from n - 1 + p objects--that is, the number of (n - 1)-membered subsets of an (n - 1 + p)-membered set.]

2. $C(4 + 6 - 1, 6 - 1) = C(9, 5) = 126$

3. 84 [or: $C(6 + 4 - 1, 4 - 1)$]

4. 10 [Put one penny in each pocket. Then count the number of ways of distributing the remaining 2 pennies among 4 pockets. This is $C(2 + 4 - 1, 4 - 1)$.]

5. 84 [Selecting p things of n kinds amounts to selecting p things from n boxes. This is just the reverse operation to distributing p things (of the same kind) among n boxes. So, the number of ways of selecting p things of n kinds is exactly the number of ways of distributing p things of the same kind in n boxes--$C(p + n - 1, n - 1)$. Exercise 5 is, essentially, a restatement of Exercise 3.]

6. 10 [This is, essentially, a restatement of Exercise 4.]

The solution of Milton's problem which is carried out in Part K depends on recognizing that one way in which Milton can choose a 9-block route is to decide at the beginning of each of 9 blocks whether to go west or south. Since, in order to arrive at Z.H.S., it is both necessary and sufficient that he walk exactly 4 blocks south [and 5 blocks west], the number of 9-block routes is the number of ways in which he can choose 4 of his 9 decisions to be his decisions to walk south. So, the number of 9-block routes is the number of 4-membered subsets of a 9-membered set--C(9, 4). The solution is completed by finding, from the table on page 7-73, that C(9, 4) = 126.

The alternative solution of Milton's problem which is suggested in Part L depends on recognizing a <u>different</u> way in which Milton can go about choosing a 9-block route. Since he has to walk a distance of 4 blocks south and a distance of 5 blocks west, it follows that any blocks which he walks on a given north-south street must be consecutive--he can never return to such a street once he has left it. So, his route is determined once he decides, for each north-south street, how many blocks he will walk south on it. Since he must walk south for a total of 4 blocks, and since he has available to him 6 north-south streets, the number of 9-block routes is the number of ways one can distribute 4 things of the same kind into 6 boxes--call it 'D(6, 4)'. In order to complete this solution of Milton's problem one needs to know how to compute values of the function D, where D(n, p) is the number of ways of distributing p things of the same kind among n boxes.

$$\forall_n \forall_p \, D(n, p) = C(p + n - 1, n - 1)$$

Hence, D(6, 4) = C(4 + 6 - 1, 5) = C(9, 5), and, by the table on page 7-73, C(9, 5) = 126.

[Another discussion of problems like Milton's is contained in the delightful article "Square Circles" by Francis Scheid in the May 1961 issue of <u>The Mathematics Teacher</u>.]

*

Answers for Part L.

1. $\forall_n \forall_p \, D(n, p) = C(p + n - 1, n - 1)$ [or: $= C(p + n - 1, p)$]

   [As suggested in the hint, the number D(n, p) of ways of distributing p things among n boxes is the number of ways of inserting n - 1 partitions into a row of p things. Once one has made such an insertion, one has a row of n - 1 + p objects, n - 1 of which are partitions.

As indicated, pages 7-76 through 7-79 are optional. They have two purposes. First, they indicate how one might organize one's knowledge of the arithmetic of the positive integers independently of his general knowledge of the real numbers--as formulated in, say, our first eleven basic principles. Second, the exercises have, like Exercise 1 of Part B on pages 7-58 and 7-59, been designed to help students learn how to discover proofs--by first concentrating on the key steps and then filling in the details.

*

The use, suggested on page 7-76, of (I_1^+) - (I_6^+) as basic principles for the positive integers is very close to the procedure used by Peano to characterize the arithmetic of the "natural numbers" [see, for example, the paper by Henkin referred to in the COMMENTARY for page 7-62, and the book <u>The Foundations of Analysis</u> by Landau [New York: Chelsea Publishing Company, 1951], pages 1-18].

Following Peano, we shall use the symbol '′' to denote the function [operation] whose value for any positive integer is the next greater positive integer. So, for example, '1′' is a numeral for 2, '1″' for 3, etc. Then one could take as basic principles for the positive integers:

(P_1) $\quad \forall_n\ n' \neq 1$

(P_2) $\quad \forall_m \forall_n\ [m' = n' \Longrightarrow m = n]$

(P_3) $\quad \forall_S\ [(1 \in S \text{ and } \forall_n\ [n \in S \Longrightarrow n' \in S]) \Longrightarrow \forall_n\ n \in S]$

$$(P_4) \quad \begin{cases} \forall_m\ m + 1 = m' \\ \forall_m \forall_n\ m + n' = (m + n)' \end{cases}$$

$$(P_5) \quad \begin{cases} \forall_m\ m \cdot 1 = m \\ \forall_m \forall_n\ m \cdot n' = m \cdot n + m \end{cases}$$

Of these, (P_1) and (P_2) correspond with (I_6^+) of page 7-76. (P_2) is related to the cancellation principle for addition [Theorem 7] and, in a development based on (P_1) - (P_5), is used [together with (P_4)] in the inductive proof of this theorem. More directly, (P_2) says that the successor function, ′, has an inverse. (P_1) and (P_2) together imply that the successor

function maps the set of positive integers in a (1 - 1) way on one of its proper subsets--that is, there are infinitely many positive integers.

(P_3) is a principle of mathematical induction--analogous to (I^+_3), and (P_4) and (P_5) are recursive definitions of addition and multiplication--analogous to (I^+_4) and (I^+_5) on page 7-76.

From (P_1) - (P_5) one can derive, for example, the commutative and associative principles for addition and multiplication of positive integers, the dpma [for positive integers] and the cancellation principles for addition and multiplication of positive integers. [The proofs of the commutative and associative principles for addition are similar to those given in the exercises on pages 7-77 through 7-79. For proofs of these and the other principles mentioned above, see the previously cited text by Landau.]

*

The procedure outlined above makes no use of an analogue, say:

(P_0) $\quad\quad 1 \in I^+ \text{ and } \forall_n\, n' \in I^+,$

of (I^+_1) and (I^+_2). The procedure described is similar to that of Unit 2 where, since we are dealing only with real numbers, it is unnecessary to introduce a name for the set of real numbers, or to adopt basic principles which say that 0 and 1 are real numbers and that the set of real numbers is closed with respect to addition, multiplication, etc. So, in the system based on (P_1) - (P_5), there is no need for a name for the set of all positive integers or for statements like (P_0) and Theorems 102 and 103. This situation should be contrasted with that in the present unit, where we are engaged in distinguishing the positive integers as real numbers of a special kind. To do this, we need a name for the set of all positive integers and we need the basic principles (I^+_1) and (I^+_2)

which say that certain real numbers are, more particularly, positive integers. And, we do need to prove closure theorems such as Theorems 102 and 103.

In order not to disrupt the students' understanding of the development in the present unit, the exercises on pages 7-77 through 7-79 are based on (I^+_1) - (I^+_4) [and Theorem 102] rather than on (P_1) - (P_4).

*

As shown by Henkin and by Landau [in the previously cited works] it is, in a sense, not even necessary to adopt (P_4) and (P_5) as basic principles. For it can be proved, using only (P_1) - (P_3) [and some basic set theory], that there exist unique operations satisfying (P_4) and (P_5). The interpretation of this result, as it applies to the dispensability of (P_4) and (P_5), involves subtleties, concerning both the role of definitions in deductive theories and the role of set theory in the foundations of mathematics, into which we cannot go at this time.

☆3. Step (1) is the theorem, '$\forall_n\ 1 + n = n + 1$', of Exercise 1 on page 7-77. Step (17) is the theorem to be proved, '$\forall_m \forall_n\ m + n = n + m$', and step (16), since it is the conclusion of the inductive step in the proof, is '$\forall_m [\forall_n\ m + n = n + m \Longrightarrow \forall_n\ (m + 1) + n = n + (m + 1)]$'. Step (15) is '$\forall_n\ q + n = n + q \Longrightarrow \forall_n\ (q + 1) + n = n + (q + 1)$', (14) is '$\forall_n\ (q + 1) + n = n + (q + 1)$', and (13) is '$(q + 1) + p = p + (q + 1)$'. Evidently, the steps preceding (14) form a test-pattern for (14) and as the comment for (7) appears to be '[(1)]', step (1) is included in this test-pattern. So, the comment for (14) is '[(1) - (13)]'. Steps (1) - (15), then, form a test-pattern for (16), and, of course, the assumption which is discharged at (15) is the inductive hypothesis, (2). Going up to (6), this is, by virtue of (I^+_1), an instance of the theorem '$\forall_m \forall_n \forall_p\ m + (n + p) = (m + n) + p$' of Exercise 2. So, the latter and (I^+_1) are steps (4) and (5). From (6) and (7) follows '$q + (p + 1) = (q + 1) + p$', and this latter is likely to be step (8). To get from (8) to (13) one needs to obtain, first, '$q + (p + 1) = (q + p) + 1$'. From this and (8) follows '$(q + 1) + p = (q + p) + 1$'. From this [(10)] and (3) follows '$(q + 1) + p = (p + q) + 1$'. From this [(11)] and a consequence, '$p + (q + 1) = (p + q) + 1$', of (4) and (5) follows '$(q + 1) + p = p + (q + 1)$', which is (13).

*

Note the two ways, illustrated in Exercises 2 and 3, of proving generalizations which contain more than one quantifier.

In Exercise 2, induction is used to obtain a test-pattern for sentences of the form '$\forall_p\ m + (n + p) = (m + n) + p$'. The theorem '$\forall_m \forall_n \forall_p\ m + (n + p) = (m + n) + p$' then follows by the test-pattern principle.

In Exercise 3 induction "on 'm'" is used directly to prove '$\forall_m [\forall_n\ m + n = n + m]$'. The test-pattern principle is used in step (14) to bring in '$\forall_n$'. And induction is also used in part (i) [since step (1) is proved by induction in Exercise 1 on page 7-77].

Correction. On page 7-77, line 5 should end:

$$--- \Longrightarrow n + 1 \in S]) \Longrightarrow \forall_n\, n \in S]$$

Answers for Exercises.

☆1. Step (9) must, of course, be '$1 + p = p + 1 \Longrightarrow 1 + (p + 1) = (p + 1) + 1$', and it must follow from (8) by conditionalizing, resulting in the discharge of (4). So, the comment for (9) is '[(8); *(4)]', step (8) is '$1 + (p + 1) = (p + 1) + 1$', and step (4), for the reason just given, as well as because it is the inductive hypothesis, is '$1 + p = p + 1$'. As a check, (8) does follow from (4) and (7) by the substitution rule for equations. Now, (7) is an instance of (I^+_4) by virtue of (I^+_1). So, (5) is (I^+_4) and (6) is (I^+_1) [or vice versa]. Step (1) must be the logical principle '$\forall_m\, m + 1 = m + 1$' and step (2) is the '$1 \in I^+$' which is needed to validate the assertion that (3) is an instance of (1). Step (11) is, of course, '$\forall_n\, 1 + n = n + 1$', and the comment is '[(3), (10), PMI]'.

☆2. Looking at (2) and (14), line (15) must be '$\forall_p\, m + (n + p) = (m + n) + p$ [(2), (14), PMI]' and step (16) must be '$\forall_m \forall_n \forall_p\, m + (n+p) = (m+n)+p$'. [As in the answer for Exercise 1 of Part A on page 7-58, the bar between lines (15) and (16) indicates that step (15) completes the inductive part of the proof.] Line (13) must be '$m + (n+q) = (m+n) + q \Longrightarrow m + (n + [q + 1]) = (m + n) + [q + 1]$ [(12), *(3)]' and step (12) must be '$m + (n + [q + 1]) = (m + n) + [q + 1]$'. Step (11) follows from (1) [which is '$\forall_m \forall_n\, m + (n + 1) = (m + n) + 1$'] and '$m + n \in I^+$'. So, the latter is step (10). Like (5), (10) is an instance of '$\forall_m \forall_n\, m+n \in I^+$'. So, this last, Theorem 102, is step (4). Step (12) follows from (11) and '$m + (n+[q+1]) = [(m+n) + q] + 1$'. So, the latter, which follows from (3) and (8), is step (9). Step (8) follows from (6) and '$n+(q+1) = (n + q) + 1$', an instance of (1). So, this is step (7).

The comment for step (2) is '[(1)]'; for (3), '[inductive hypothesis]'; for (4), '[Theorem 102]'; for (5), '[(4)]'; for (6), '[(1), (5)]'; for (7), '[(1)]'; for (8), '[(6), (7)]'; for (9), '[(3), (8)]'; for (10), '[(4)]'; for (11), '[(1), (10)]'; and for (12), '[(9), (11)]'.

Answers for Miscellaneous Exercises.

[easy: 1, 3, 4, 6, 7, 9, 10, 12, 13, 14, 16, 19, 24, 27, 28, 31, 32, 33, 34, 35; medium: 2, 5, 8, 11, 17, 18, 20, 21, 23, 26, 29, 30, 36; hard: 15, 22, 25]

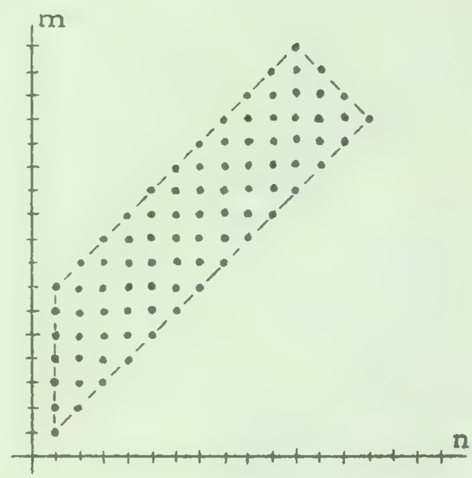
1. (A)

2. $2x/7$

3. [There are 86 solutions of the system:

$$\begin{cases} m + n \le 28 \\ m - n \le 6 \\ n \le m \end{cases}$$

Students may enjoy graphing the solution set.]



4. (1, 1)

5. 4 : 5

6. (a) $\frac{ab}{2x}$ (b) $\frac{7}{ab^2}$ (c) $\frac{15}{a - b}$

7. $x = ab^2$ [$a \ne 0$, $b \ge 0$] [Note, particularly, the restriction '$b \ge 0$'. Point out to students that, for example, '$\sqrt{x} = -2$' had no solution (and, so, is not equivalent to '$x = (-2)^2$'). See page 3-131 of Unit 3.]

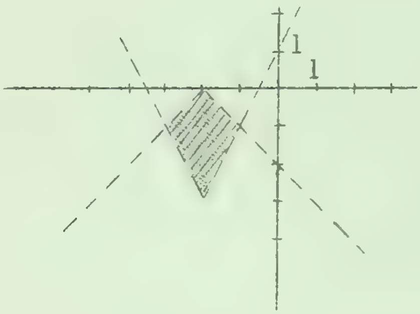
8. 7

9. 90 pounds

[Since there will be a need for mensuration formulas in many of the miscellaneous exercises in this and subsequent units, it may be a good idea to have students collect those studied in earlier grades and list them in a convenient place in their textbooks.]

10. $\frac{9}{2}$, or: $-\frac{9}{2}$ [two answers]

11.



12. $x + y + z$ years old

13. (E)

14. (a) $3a^2 + 5c$

(b) $5k^2 + 6k - 5m^2$

(c) $-10mn + 2n^3 - m^2$

15. 180   16. 13   17. 30   18. 5   19. $(-1, 3)$

20. 1 : 4 [$K(\triangle EFG) = \frac{1}{2}K(\triangle EFA)$, $K(\triangle EFA) = \frac{1}{2}K(\square EFDA)$]

21. $t = \dfrac{100A(V - V_0)}{V_0}$

[The answer given for Exercise 21 is correct if one assumes that all five variable quantities named in the exercise have the same domain and that 0 is not a value of V_0, $V_{100} - V_0$, or A. See page 5-112 <u>et seq</u>. of Unit 5.]

22. 6

23. 20 gallons

24. (a) $\dfrac{2(6m + 1)}{m^2 - 1}$ (b) $\dfrac{4 + t}{(4 - t)(5 - t)}$ (c) $\dfrac{7}{2x - y}$

25. 2 [Let a_0 be the number of bags which the merchant started out with and, for each $n \le 13$, let a_n be the number of bags he has left after passing n gates. Then, $\forall_{n \le 13}\ a_n = \frac{1}{2}a_{n-1} + 1$. So, $\forall_{n \le 13}\ a_n - 2 = \frac{1}{2}(a_{n-1} - 2)$. Hence, $a_{13} - 2 = \dfrac{a_0 - 2}{2^{13}}$. Since $a_{13} = a_0$, it follows that $a_0 - 2 = 0$. So, the merchant started out with 2 bags.]

26. $70

27. $\frac{1}{20}$

28. (a) $\frac{7a}{2}$ (b) $\frac{1}{t}$

29. $16\frac{4}{11}$ miles

30. $K = \frac{(\ell + w)^2}{\pi}$

31. 1

32. $\frac{62.008}{0.5749}$ [approx. 107.86]

33. (a) 2 (b) 1 (c) 30 (d) $\frac{1}{3}$

34. $\{(x, y): 7y = 2x - 14\}$

35. 96¢

36. $\frac{2\pi x^2}{y}$

Correction. On page 7-85, line 14 should end: $--- 1 - 1 \in I^+.$

Theorems 104 [page 7-84], 106 [page 7-86], and 108 [page 7-88] are very important. To gain at least minimal understanding of these theorems students should study pages 7-84, 7-87, and 7-89 thoroughly. However, if pressed for time, omit the proof, on page 7-85, of Theorem 105 and that, on page 7-88, of Theorem 108. Also, some [even all] of the exercises on pages 7-86 and 7-90, 91 may be omitted, if time presses.

*

Given a positive number, its half is a smaller positive number [so, there is no least positive number].

Given two positive numbers, their average is a positive number which is between them [so, between any two positive numbers there is a third].

[For proofs of the statements above, see answer for Part ☆F on page 7-86.]

*

The explanation asked for at the bottom of page 7-84 might go as follows: Suppose that $n > m$. Then, by Theorem 105, $n - m \in I^+$. Hence, by Theorem 104, $n - m \geq 1$. So [assuming Theorems 104 and 105], if $n > m$ then $n - m \geq 1$.

*

The first of the two explanations asked for on page 7-85 is, of course, that Theorem 87 ['$\forall_x\, x \not> x$'] is, trivially, equivalent to '$\forall_x \forall_y [x = y \Rightarrow x \not> y]$' [see COMMENTARY for page 7-33]. By contraposition, the latter is equivalent to '$\forall_x \forall_y [x > y \Rightarrow x \neq y]$'.

The second explanation is that (a) follows from (b) by virtue of the rule for denying an alternative [see the COMMENTARY for page 7-34].

*

The test-pattern for part (ii) of the inductive proof of (b) may be diagrammed as follows:

$$\cfrac{\cfrac{r}{s \text{ or } r}\quad \text{[a basic rule for 'or']}}{t \Rightarrow s \text{ or } r}\quad \text{[conditionalizing]}$$

[r: $(p + 1) - 1 \in I^+$; s: $p + 1 = 1$; t: $p = 1$ or $p - 1 \in I^+$]

Answers for Part A.

1. (i) Since $1 = 1$, it follows that $1 \geq 1$.

   (ii) Suppose that $p \geq 1$. By Theorem 90, $p + 1 > p$ and, so, by Theorem 92, it follows that $p + 1 > 1$. Hence, $p + 1 \geq 1$. Consequently, $\forall_n [n \geq 1 \Longrightarrow n + 1 \geq 1]$.

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\, n \geq 1$.

2. By Theorem 104, $p \geq 1$ and, by Theorem 90, $p + 1 > p$. So, by Theorem 92, $p + 1 > 1$. Consequently, $\forall_n\, n + 1 > 1$.

*

Answers for Part B.

1. By (a), if $p \neq 1$ then $p - 1 \in I^+$. Since either $p = 1$ or $p \neq 1$, it follows [by the rule of the dilemma] that $p = 1$ or $p - 1 \in I^+$. Consequently [assuming (a)], $\forall_n [n = 1 \text{ or } n - 1 \in I^+]$.

   By (b), either $p = 1$ or $p - 1 \in I^+$. Suppose that $p \neq 1$. Then [by the rule for denying an alternative], it follows that $p - 1 \in I^+$. Hence, if $p \neq 1$ then $p - 1 \in I^+$. Consequently [assuming (b)], $\forall_n [n \neq 1 \Longrightarrow n - 1 \in I^+]$.

2. Suppose that $p \neq 1$. Since, by (a), if $p \neq 1$ then $p - 1 \in I^+$, it follows that $p - 1 \in I^+$. So, by Theorem 104, $p - 1 \geq 1$, and by Theorems 86d and 32, $p \geq 1 + 1$. Hence, if $p \neq 1$ then $p \geq 2$. Consequently, $\forall_n [n \neq 1 \Longrightarrow n \geq 2]$.

3. By Exercise 2, if $p \neq 1$ then $p \geq 2$. By Theorem 88, $p \geq 2$ if and only if $2 \not> p$. Hence, if $p \neq 1$ then $2 \not> p$--that is, if $p < 2$ then $p = 1$. Consequently, $\forall_n [n < 2 \Longrightarrow n = 1]$--the only positive integer less than 2 is 1.

*

Answers for Part C.

1. Suppose that $p > q$. Since, by Theorem 105, if $p > q$ then $p - q \in I^+$, it follows that $p - q \in I^+$. Hence, by Theorem 104, $p - q \geq 1$. So, by Theorems 86d and 32, $p \geq 1 + q = q + 1$. Hence, if $p > q$ then $p \geq q + 1$. Consequently, $\forall_m \forall_n [n > m \Longrightarrow n \geq m + 1]$.

2. Suppose that $p \geq q + 1$. Since, by Theorem 90, $q + 1 > q$, it follows [by a theorem like Theorem 92] that $p > q$. Hence, if $p \geq q + 1$ then $p > q$. Consequently, $\forall_m \forall_n [n \geq m + 1 \Longrightarrow n > m]$.

*

Answers for Part D.

1. Suppose that $a \in I^+$. Since $5 \in I^+$, it follows from Theorem 106 that if $a > 5$ then $a \geq 5 + 1 = 6$. So, by Theorem 88, if $a > 5$ then $a \not< 6$ --that is, not both $a > 5$ and $a < 6$ [see COMMENTARY for page 7-35]. Hence, if $a \in I^+$ then not $(5 < a < 6)$--that is, if $5 < a < 6$ then $a \notin I^+$. Consequently, $\forall_x [5 < x < 6 \Longrightarrow x \notin I^+]$.

2. [The answer for Exercise 1 is, practically, a test-pattern for the theorem in question. From the second sentence of that answer, delete the beginning, 'Since $5 \in I^+$,' and the end, '$= 6$'. Replace each '5' by an 'n' and each '6' by an '$n + 1$'. Insert '$\forall_n$' before '$\forall_x$'.]

*

Answers for Part E.

1. By Theorem 104, $p \geq 1$. So, by Theorem 88, $p \not< 1$. Consequently, $\forall_n\, n \not< 1$.

2. By Theorem 106, $p \geq q + 1$ if and only if $p > q$. Hence, $p \not\geq q + 1$ if and only if $p \not> q$. So, by Theorem 88, $p < q + 1$ if and only if $q \leq p$. Consequently, $\forall_m \forall_n [n < m + 1 \Longleftrightarrow n \leq m]$.

*

Answer for Part ☆F.

[The strategy is suggested in the second remark in the COMMENTARY for page 7-84.]

Since $2 > 0$, it follows from Theorem 99b that $\frac{1}{2} > 0$. So, by Theorem 86e, $\forall_x [x > 0 \Longrightarrow x \cdot \frac{1}{2} > 0]$. In particular [using Theorem 84], $\forall_x \forall_y [x > y \Longrightarrow (x - y)/2 > 0]$.

Since $\forall_x \forall_y\, x - (x + y)/2 = (x - y)/2 = (x + y)/2 - y$, it follows [again using Theorem 84] from the result just obtained that $\forall_x \forall_y [x > y \Longrightarrow x > (x + y)/2 > y]$.

In particular, $\forall_{x > 0}\, x > x/2 > 0$ [this is (1)] and (2).

lines 30-32: Since a least member of a set is, in particular, a lower bound of the set, it follows that a set which has a least member has a lower bound. But, as the example of P shows, a set may have lower bounds yet not have a least member.

lines 33-35: A lower bound of a set is, also, a lower bound of each of its subsets. So, if a set has a lower bound then so does each of its subsets. However, a set which has a least member may have subsets which do not have least members. [For example, $P \subseteq P \cup \{0\}$, $P \cup \{0\}$ has a least member, but P does not.] And, a subset of a set may have a least member [or a lower bound] even though the set itself does not. [For example, $\{n: n > 6\} \subseteq \{x: x > 6\}$, the first has a least member, but the second does not.]

*

In summary [and extension], there are sets of real numbers which do <u>not</u> have lower bounds [p.e., $\{x: x < 0\}$]. [To answer a question which may arise, each number is a lower bound of $\emptyset$.] If a nonempty set of real numbers <u>does</u> have a lower bound then [although this has not been stressed, and cannot be proved until a later unit], it has a unique <u>greatest</u> lower bound, and its lower bounds are just the numbers less than or equal to this greatest lower bound. Such a set may or may not have a least member. It does if and only if its greatest lower bound belongs to the set.

Answers for questions in text.

lines 5, 6: 0 is less than or equal to each positive number. Each negative number is less than or equal to each positive number. So, there are infinitely many numbers each of which is less than or equal to all the members of P--that is, P has infinitely many lower bounds.

line 9: 0 is the greatest of the lower bounds of P.

lines 10, 11: K does have a least member, the number 3. Each number less than or equal to 3 is a lower bound of K [and no other number is a lower bound of K].

lines 12, 13: $\{x: x > 6\}$ does not have a least member [if a belongs to this set then $(a + 6)/2$ is a smaller number which also belongs]. The lower bounds of $\{x: x > 6\}$ are just the numbers less than or equal to 6.

lines 14, 15: Since $\{n: n > 6\} \subseteq \{x: x > 6\}$, each lower bound of the latter is, also, a lower bound of the former. Also, $\{n: n > 6\}$ has 7 as its least member. [Hence, each number less than or equal to 7 is a lower bound of $\{n: n > 6\}$.]

lines 16-18: The set of negative integers does not have a least member [if j is any negative integer, $j - 1$ is a smaller one]. The set of negative integers has no lower bound--equivalently, the set of positive integers has no upper bound. [To prove either of these equivalent statements requires a new basic principle--the cofinality principle on page 7-89.]

lines 26, 27: Suppose that a is a lower bound of S and that $b < a$. It follows that, for c in S, $a \leq c$. So, by Theorem 92, $b < c$. Hence, for c in S, $b \leq c$--that is, b is a lower bound of S. Consequently, each number which is less than some lower bound of a given set is, also, a lower bound of the set.

lines 28, 29: Suppose that a is a least member of S and that b is a least member of S. Then, $a \in S$, $b \in S$, and, for c in S, $a \leq c$ and $b \leq c$. Hence, $a \leq b$ and $b \leq a$. So, by Theorem 93, $a = b$. Consequently, a set has <u>at most one</u> least member.

The least number theorem is closely related to (I^+_3). As is illustrated by the proof in Unit 4 that there is no rational number whose square is 2, Theorem 108 is a very powerful means of proof. In fact, with the help of some "smaller" theorems, it is easy to derive (I^+_3) from Theorem 108. The most direct procedure is to use, besides Theorem 108, the theorems:

(1) $\forall_{n \neq 1}\ n - 1 \in I^+$

(2) $\forall_{n \neq 1}\ (n - 1) + 1 = n$

and: (3) $\forall_{n \neq 1}\ n \not\leq n - 1$

Of these, (1) is a reformulation of (a) on page 7-85, (2) is a special case of Theorem 32, and (3) is a consequence of (2) and Theorems 88 and 90. The derivation of (I^+_3) from Theorem 108 and (1), (2), and (3) goes as follows:

> Suppose that $1 \in S$ and that $\forall_n\ [n \in S \Longrightarrow n + 1 \in S]$. Suppose, further, that there is a positive integer which does not belong to S. From this it follows, by Theorem 108, that there is a least such positive integer, say p. Then, $p \notin S$ and $\forall_n\ [n \notin S \Longrightarrow p \leq n]$. Since $p \notin S$ and $1 \in S$, it follows that $p \neq 1$. Consequently, by (1), $p - 1 \in I^+$ and, by (3), $p \not\leq p - 1$. So, since $\forall_n\ [n \notin S \Longrightarrow p \leq n]$, it follows that $p - 1 \in S$. Hence, since $\forall_n\ [n \in S \Longrightarrow n + 1 \in S]$, it follows that $(p - 1) + 1 \in S$. Consequently, by (2), $p \in S$. But, $p \notin S$. Hence, there is no positive integer which does not belong to S--$\forall_n\ n \in S$.

The rather trivial explanation asked for near the bottom of page 7-89 is most easily given via a column proof:

(1)	$\forall_x \exists_n\ n > x$	[cofinality principle]
(2)	$\exists_n\ n > -a$	[(1)]
(3)	$\forall_x \exists_n\ n > -x$	[(1), (2)]

The next two generalizations [occurring in the next-to-bottom line of page 7-89] follow, in succession, by virtue of the biconditionals '$n > -x \Longleftrightarrow x > -n$' [Theorems 17 and 94] and '$x > -n \Longleftrightarrow x \not\leq -n$' [Theorem 88].

The word 'cofinal' [sometimes 'confinal'] suggests that positive integers "go out as far as the real numbers do".

*

'$\exists_n$' is read as 'there is an **n** [or: there is a positive integer n] such that'.

*

Although the basic principles we have adopted up to now are sufficient to characterize the set of positive integers, they do not completely describe how the positive integers are distributed among the real numbers. The cofinality principle takes care of this question. It says that, given any real number [no matter how large], there is a positive integer which is greater.

The cofinality principle is a special case of the Archimedean principle. The latter says that, given two real numbers, one of which is positive, there is a positive integral multiple of the latter which is greater than the other--$\forall_{x > 0} \forall_y \exists_n\, xn > y$. Since this latter principle can be rewritten as '$\forall_{x > 0} \forall_y \exists_n\, n > \frac{y}{x}$', it is not difficult to see that, once the cofinality principle has been accepted as a basic principle, the Archimedean principle is a theorem.

In a later unit we shall adopt a completeness principle:

Each nonempty set which has an upper bound has a least upper bound.

Since I^+ is nonempty, it follows from this, and the assumption that I^+ has an upper bound, that I^+ has a least upper bound. Suppose that a is the least upper bound of I^+. Then $a - 1$ is not an upper bound of I^+ and there is a positive integer, say p, such that $p > a - 1$. But, then, $p + 1 > a$. Since $p + 1 \in I^+$, it follows that a is not an upper bound of I^+. From this contradiction it follows that I^+ has no upper bound--$\forall_x \exists_n\, n > x$. So, once the completeness principle has been adopted, the cofinality principle becomes a theorem and can be dropped from the basic principles.

*

Answers for Part A.

[For each exercise, Exercises 5 and ☆8 excepted, there are many sets which satisfy the given conditions. (There is none which satisfies the conditions in Exercise 5, and $\emptyset$ is the only set which satisfies the conditions of Exercise ☆8.)]

1. $\{x: 1 < x \leq 2\}$, $\{1, 59/2\}$, $\emptyset$, the union of finitely many sets each of which satisfies the given conditions, any subset of a set which satisfies the given conditions, ...

2. P, I^+, $\{x: x > -10^6\}$, the union of finitely many sets some (at least one) of which satisfy the given conditions while the rest (if any) satisfy the conditions of Exercise 1, ...

3. the set of all negative numbers, the set of all negative integers, $\{x: x < 10^6\}$, any set S such that $\{x: -x \in S\}$ satisfies the conditions of Exercise 2, ...

4. the set of all integers, any set one of whose subsets satisfies the given conditions, any set which is the union of two sets one of which satisfies the conditions of Exercise 2 and the other those of Exercise 3, ...

5. [There is no such set. If a set has a greatest member then, by definition, this greatest member is an upper bound of the set.]

6. $\{x: 1 < x \leq 2\}$, $\{x: x \leq 526\}$, $\{n: n < 73\}$, (trivially) any set which has a greatest member, ...

7. $\{x: 1 < x < 2\}$, the set of negative rational numbers, ... [By Theorem 115 on page 7-100, there is no set of integers which satisfies these conditions.]

☆8. $\emptyset$ [There is no other set which satisfies these conditions.]

*

Answers for Part B.

1. For $a > 0$, it follows that $a \neq 0$ and, by the cofinality principle, there is a positive integer greater than $\frac{1}{a}$. Let p be such a positive integer. Since, for $a > 0$, $\frac{1}{a} > 0$ and $\frac{1}{1/a} = a$, it follows, by Theorem 100, that $a > \frac{1}{p}$. So, for $a > 0$, $\exists_n\, a > \frac{1}{n}$. Consequently, $\forall_{x>0} \exists_n\, x > \frac{1}{n}$. [That is, given any positive number, no matter how small, there is a positive integer whose reciprocal is smaller.]

2. (a) yes; yes [The upper bounds of $\mathcal{R}_f$ are the numbers greater than or equal to 1; its lower bounds are the nonpositive numbers.]

   (b) yes; no [The greatest member of $\mathcal{R}_f$ is 1; 0 is the greatest lower bound of $\mathcal{R}_f$ but does not belong to this set.]

3. (a) yes; yes [The least upper bound of $\mathcal{R}_f$ is $\frac{1}{2}$; its greatest lower bound is -1.]

   (b) yes; yes [The greatest member of $\mathcal{R}_f$ is $\frac{1}{2}$; its least member is -1.]

4. (a) yes; yes [The least upper bound of $\mathcal{R}_f$ is 1; its greatest lower bound is 0.]

   (b) no; yes [The least member of $\mathcal{R}_f$ is 0.]

*

Answers for Part C.

1. (a) If x is a positive rational number then [by the definition of 'rational number'] $\{n\colon \exists_m\, xn = m\} \neq \emptyset$. So, Theorem 108 tells you that $\{n\colon \exists_m\, xn = m\}$ has a least member.

   (b) If x is an irrational number then, by definition, $\{n\colon \exists_m\, xn = m\} = \emptyset$. So, Theorem 108 tells you nothing about this set. [But, since it is $\emptyset$, there is not much to be said about it, anyway.]

☆2. Suppose that $m > n$. It follows, by Theorem 105, that $m - n \in I^+$. From this, and since $m - n = m - n1$, it follows that $m - n \in \{q: \exists_p\ q = m - np\}$. Hence, this set is nonempty and, by Theorem 108, has a least member, say q_o. Since $q_o \in \{q: \exists_p\ q = m - np\}$, there is a positive integer, say p_o, such that $q_o = m - np_o$. Since $q_o \in I^+$, it follows that $q_o > 0$. Suppose that $q_o > n$. By Theorem 105, $q_o - n \in I^+$. Since $q_o - n = (m - np_o) - n = m - n(p_o + 1)$, and since, by (I^+_2), $p_o + 1 \in I^+$, it follows that $q_o - n \in \{q: \exists_p\ q = m - np\}$. But this is impossible. For, since $n > 0$, it follows that $q_o - n < q_o$, and q_o is the <u>least</u> member of the set in question. Hence, $q_o \not> n$, and, by Theorem 88, $q_o \leq n$. Consequently, $\exists_p\ 0 < m - np \leq n$.

*

Answers for Part D.

1. (a) Suppose that $f_p = g_p$--that is, that $p(p + 1) = (p + \frac{1}{2})^2$. Since $f_{p+1} = (p + 1)(p + 2) = p(p + 1) + 2(p + 1)$, it follows that $f_{p+1} = (p + \frac{1}{2})^2 + 2(p + 1) = p^2 + 3p + \frac{9}{4} = (p + \frac{3}{2})^2 = [(p + 1) + \frac{1}{2}]^2 = g_{p+1}$. So, if $f_p = g_p$ then $f_{p+1} = g_{p+1}$. Consequently, $\forall_n [f_n = g_n \Rightarrow f_{n+1} = g_{n+1}]$.

   (b) no $[f_4 = 20 \neq 81/4 = g_4]$; no; Both parts of an inductive proof are essential in establishing a generalization.

(n + 1)-membered set] has $C_n + C_n$ subsets. Consequently, $\forall_n C_{n+1} = 2 \cdot C_n$.] [Some students may suggest, correctly, that $\forall_n C_n = 2^n$. To derive this result from the answer for part (a) they would need to use the definition: $2^1 = 2$ and $\forall_n 2^{n+1} = 2 \cdot 2^n$]

(b) Let S be a nonempty set and let e_o be a member of S. Let S_o be the subset consisting of those members of S other than e_o. Each odd-membered subset of S is either an odd-membered subset of S_o or the union of $\{e_o\}$ and an even-membered subset of S_o. Hence, there are exactly as many odd-membered subsets of S as there are subsets [odd- or even-membered] of S_o. Similarly, there are exactly as many even-membered subsets of S as there are subsets of S_o. So, S has the same number of odd-membered subsets as it has even-membered subsets. [So, for each n, an (n + 1)-membered set has 2^n odd-membered subsets and 2^n even-membered subsets. Notice that this result says nothing about $\emptyset$ which, as a matter of fact, has no odd-membered subsets and one even-membered subset.]
[The check asked for amounts to adding alternate entries in a row in the table on page 7-73. For example, the numbers given in the 8-row of the table are 1, 8, 28, 56, 70, 56, 28, 8, 1. So, the number of odd-membered subsets of an 8-membered set is 8 + 56 + 56 + 8, or 128, and the number of even-membered subsets is 1 + 28 + 70 + 28 + 1, or 128.]

(c) 2048 [or: 2^{11}] [A route is determined uniquely by which of the eleven sections of, say, the right side of the ladder are covered by the route. So, the number of routes is the number of subsets of an 11-membered set.]

> Correction. On page 7-91, line 3 should read:
>
> (a) Give a counter-example for (*).

Answers for Part D [continued].

2. (a) (2, 1) [$1 \in I^+$, $2 \in I^+$, $1 \leq 2$, but $1 \neq 2$]

   [Strictly, 2 is a counter-example for the generalization '$\forall_m (\forall_n [n \leq m \Longrightarrow n = m])$', since $2 \in I^+$ and '$\forall_n [n \leq 2 \Longrightarrow n = 2]$' is false. This last generalization has 1 as a counter-example, since $1 \in I^+$ and '$1 \leq 2 \Longrightarrow 1 = 2$' is false. But, it should cause no confusion to say that (2, 1) is a counter-example for the given "double generalization".]

   (b) In part (ii) ["So, from the inductive hypothesis, ..."] it is implied that '$q - 1 \leq p \Longrightarrow q - 1 = p$' is an instance of the inductive hypothesis. But, to support this claim, one needs '$q - 1 \in I^+$'. So, from this inductive hypothesis it follows only that $\forall_{n > 1} [n \leq p + 1 \Longrightarrow n = p + 1]$.

   (c) Note, in the fourth line from the bottom of page 7-85, 'So, from (i), $q - 1 \in I^+$'.

*

Answers for Miscellaneous Exercises.

[easy: 2-6, 8-10, 12-15, 18-20; medium: 7, 11, 17; hard: 1, 16]

1. (a) $C_1 = 2$, $\forall_n C_{n+1} = 2 \cdot C_n$ [or: $C_0 = 1$, $\forall_n C_n = 2 \cdot C_{n-1}$]

   [A set consisting of a single element has just two subsets--$\emptyset$, and the set itself. So, $C_1 = 2$. Using the notation of the hint, the subsets of S which do not contain e_0 are precisely the subsets of S_0, and there are C_n of these. Each subset of S which does contain e_0 is the union of $\{e_0\}$ and a subset of S_0 and, in this way, different subsets of S_0 give rise to different subsets of S. So, there are exactly as many subsets of S which contain e_0 as there are subsets of S_0--and there are C_n of these. Hence, S [an

2. (a) $\begin{cases} f_1 = 8 \\ \forall_n\, f_{n+1} = f_n + 7 \end{cases}$ (b) $\begin{cases} f_1 = 16 \\ \forall_n\, f_{n+1} = f_n + 7 \end{cases}$

(c) $\begin{cases} f_1 = -1 \\ \forall_n\, f_{n+1} = f_n - 1 \end{cases}$ (d) $\begin{cases} f_1 = 1 \\ \forall_n\, f_{n+1} = f_n - 5 \end{cases}$

(e) $\begin{cases} f_1 = 1 \\ \forall_n\, f_{n+1} = f_n + (2n + 1) \end{cases}$ (f) $\begin{cases} f_1 = -3 \\ \forall_n\, f_{n+1} = f_n + (4n + 2) \end{cases}$

[As to alternative answers, see comment on answers for Part F on page 7-69.]

3. (a) 1 (b) 1 (c) 2 (d) 1

[Of course, there are many other correct answers.]

4. 300

5. $1\frac{1}{4}$ pounds [or: $80/(\sqrt{16})^3$ pounds]

6. (a) $7s^2 + 5rs - r^2$ (b) $29x^3 - 40x^2 + 5x + 4$

(c) $-6a^2 + 5ab + 27b^2$

7. 1/11 minutes

8. $2\sqrt{6}$; 4

9. (3, −5)

> <u>Correction.</u> On page 7-93, line 9 should end:
>
> $$---,\ 3^x \cdot 9^x = \underbrace{\quad}_{\uparrow}.$$

10. $V = \dfrac{hn - E}{e}$

[The answer for Exercise 10 is correct if one assumes that all seven variable quantities mentioned in the exercise have the same domain and that 0 is not a value of e. See page 5-112 <u>et seq</u>. of Unit 5.]

11. $n = \dfrac{m + 1}{m}$ $[m \neq 0 \neq n \neq 1]$

12. 0

13. $\dfrac{5}{12}$, $\dfrac{5}{12}$

14. (C)

15. 13/8

16. 9x/16

17. 19.2 days

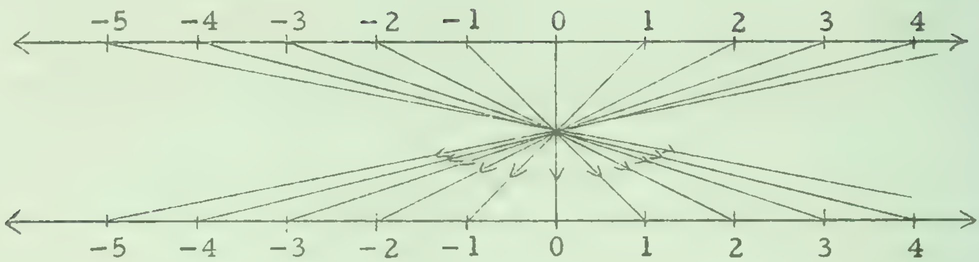
18. 4

19. (a) $\dfrac{(a + 4)(a + 5)}{a + 1}$ (b) $\dfrac{3}{5r}$

20. (1, 1)

Answers for Part B.

1.



Yes, r does have an inverse.

2. (a) $\{j: j < 1\}$ (b) $\{j: j > -5\}$ (c) $\{j: 4 < j < 5\}$

(d) $\{j: -5 < j < 6\}$ (e) $\{j: j > 8\}$ (f) $\{j: j > -k\}$

3. $\Longleftrightarrow j > i$

4. (a) By the cofinality principle, there is a positive integer, say n, such that $-a < n$. It follows, by Theorem 94 [and Theorem 17], that $-n < a$. By (I), $n \in I$ and, so, by Theorem 110a, $-n \in I$. Hence, $\exists_j\, j < a$. Consequently, $\forall_x \exists_j\, j < x$.

(b) Suppose that a is a lower bound of S. By the result just proved, there is an integer, say k, such that $k < a$. By a result whose proof was asked for in the text on page 7-87, it follows that k is a lower bound of S. So, if S has a lower bound then it has an integral lower bound.

*

The answer for part (b) of Exercise 4 tells you:

If S has a lower bound then $\exists_k \forall_{y \in S}\, k \leq y$.

Notice that in view of part (a) of Exercise 4 we have a stronger theorem:

If S has a lower bound then $\exists_k \forall_{y \in S}\, k < y$.

This stronger remark is used in the proof of Theorem 113 on page 7-98.

$a - 5 \in I$, it follows that $a - 5 \in S$. Consequently, if $a \in I$ and $a > -2$, then $\exists_i$ ($i \in S$ and $a = i - -5$)--that is, $a \in S^*$.

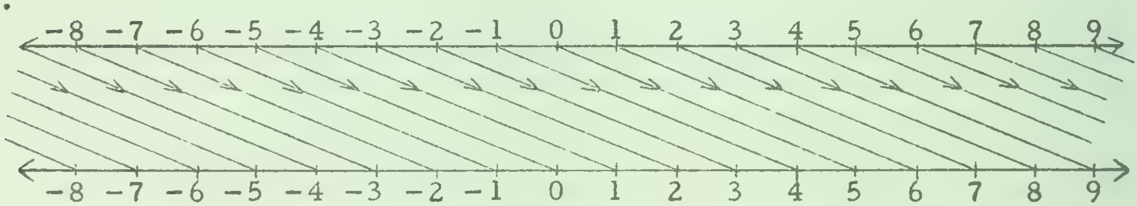
So, $S^* = \{x: x \in I \text{ and } x > -2\} = \{k: k > -2\}$.

*

Make sure that students do not confuse the notation 't_{-5}' of Exercises 2 and 3 and the 't_j's of Exercise 4, etc. with previous notations such as 'f_4' and 'g_n'. t_{-5} is a function whose arguments are integers--$t_{-5}(4)$, for example, is $4 - -5$, or 9. In other words, t_{-5} is the operation subtracting -5, restricted to integers. [The symbol 't' has been chosen to suggest the word 'translation' as, in Part B, 'r' suggests 'reflection'.]

*

2.



3. I; yes; I^+

4. (a) I (b) $t_j(i) > t_j(k)$

(c) $\forall_i \forall_k [t_j(i) > t_j(k) \Rightarrow i > k]$; $\forall_i \forall_k [t_j(i) = t_j(k) \Rightarrow i = k]$

[The property of t_j noted in part (b) says that t_j "preserves order". The second answer for part (c) says that the function t_j has an inverse; from the first answer it follows that this inverse also preserves order.]

5. (a) I^+ (b) I^+ (c) I^+

*

The message of Part A is that, for any j, $\{k: k > j\}$ can be mapped, preserving order, on I^+. That of Part B is that, for any j, $\{k: k > j\}$ can be mapped, <u>reversing</u> order, on $\{k: k < -j\}$ [and vice versa].

*

$(m + q) - (n + p) > 0$, and, by Theorem 84, $m + q > n + p$. So, $m + q > n + p$ if and only if $(m + q) - (n + p) \in I^+$. Hence, $k > j$ if and only if $(m + q) - (n + p) \in I^+$. Since $k - j = (m - n) - (p - q) = (m + q) - (n + p)$, it follows that $k > j$ if and only if $k - j \in I^+$.

*

Note two corollaries of Theorem 111:

$$\forall_k [k > 0 \Longleftrightarrow k \in I^+] \quad \text{and:} \quad \forall_j [j < 0 \Longleftrightarrow -j \in I^+]$$

*

Proof for Theorem 112: Suppose that $k + 1 > j$. It follows, by Theorem 111, that $k + 1 - j \in I^+$ and, so, by Theorem 104, that $k + 1 - j \geq 1$--that is, that $k \geq j$. Hence, if $k + 1 > j$ then $k \geq j$.

Next, suppose that $k \geq j$. Since, by Theorem 90, $k + 1 > k$, it follows, by Theorem 92, that $k + 1 > j$. Hence, if $k \geq j$ then $k + 1 > j$.

Consequently, $k + 1 > j$ if and only if $k \geq j$.

*

Answers for Part A.

1. $S^* = \{k: k > -2\}$

*

All that is required for Exercise 1 is the answer just given. However, you may wish to point out that justification of this answer depends on the fact that $-5 \in I$ and that I is closed with respect to subtraction. For example, the set obtained by subtracting a noninteger, say π, from each member of S is not $\{k: k > -7 + \pi\}$. [In fact, the former of these two sets contains no integers at all.]

By definition, $S^* = \{x: \exists_i\ i \in S \text{ and } x = i - -5\}$.

Now, suppose that $a \in S^*$. Then, by definition, there is an $i \in S$ such that $a = i - -5$. Since $-5 \in I$ and I is closed with respect to subtraction, $a \in I$. And, since $i > -7$, $a > -2$. Consequently, if $a \in S^*$ then $a \in I$ and $a > -2$.

On the other hand, suppose that $a \in I$ and $a > -2$. Then [since $5 \in I$], $a - 5 \in I$ and $a = (a - 5) - -5$. Furthermore, $a - 5 > -7$ and so, since

our usual procedure has been to make a temporary division on the occasion of each separate proof.

The second reason for distinguishing between apparent variables and real variables is to simplify the statement of the rule for universal instantiation. Briefly, one needs to do something to ensure that this rule does not justify inferring, for example, from the premiss '$\forall_x \exists_y\, y > x$', the conclusion '$\exists_y\, y > y$'--that is, inferring, from 'there is no greatest number', the conclusion 'there is a number which is greater than itself'. If one is to maintain that a universal generalization implies each of its instances he must, then, define 'instance' in such a way that '$\exists_y\, y > y$' is not an instance of '$\forall_x \exists_y\, y > x$'. The simplest procedure is to require that the symbols which are apparent variables be different from those which are real variables and to specify that a sentence is an instance of a universal generalization only if it results from removing the quantifier and replacing the apparent variable by an expression which does not itself contain any apparent variable. So, for example, one could form an instance of '$\forall_x \exists_y\, y > x$' by replacing the 'x' in '$\exists_y\, y > x$' by a real variable ['$\exists_y\, y > a$'] or by a numeral ['$\exists_y\, y > 2$'], but <u>not</u> by replacing 'x' by the apparent variable 'y' ['$\exists_y\, y > y$']. Such a specification would occur, in a formal treatment, among the rules of grammar which have been referred to in the COMMENTARY for pages 7-39 and 7-40.

*

Proof for Theorem 110c: Suppose that k and j are integers. Then, by Theorem 109, there are positive integers m, n, p, and q such that $k = m - n$ and $j = p - q$. So, $k - j = (m - n) - (p - q) = (m + q) - (n + p)$. By Theorem 102, $m + q \in I^+$ and $n + p \in I^+$. Hence, by Theorem 109, $k - j \in I$.

Proof for Theorem 110d: Suppose that k and j are integers. Then, by Theorem 109, there are positive integers m, n, p, and q such that $k = m - n$ and $j = p - q$. So, $kj = (m - n)(p - q) = (mp + nq) - (mq + np)$. By Theorems 102 and 103, $mp + nq \in I^+$ and $mq + np \in I^+$. Hence, by Theorem 109, $kj \in I$.

*

Proof for Theorem 111: Suppose that k and j are integers. Then, by Theorem 109, there are positive integers m, n, p, and q such that $k = m - n$ and $j = p - q$. Now, $k > j$ if and only if $m - n > p - q$--that is, if and only if $m + q > n + p$. By Theorem 102, $m + q \in I^+$ and $n + p \in I^+$. Hence, by Theorem 105, if $m + q > n + p$ then $(m + q) - (n + p) \in I^+$. On the other hand, if $(m + q) - (n + p) \in I^+$ then, by Theorem 101,

The set I is, of course, not closed with respect to division--$1 \in I$, $2 \in I$, and $\frac{1}{2} \notin I$.

There are several ways to prove that $\frac{1}{2} \notin I$. We shall give two such proofs--a short one which makes use of a consequence ['Each integer greater than 0 belongs to I^+'] of Theorem 111 on page 7-96, and a longer one which makes use of Theorem 109 and other previously proved theorems. Both proofs make use of the fact that $0 < \frac{1}{2} < 1$, and we begin by proving this.

> Since $2 \neq 0$, it follows by Theorem 99b that, since $2 > 0$, $\frac{1}{2} > 0$. Since $1 < 2$, it then follows that $\frac{1}{2} < 1$ [mtpi, cpm, pm1, and pq]. Consequently, $0 < \frac{1}{2} < 1$. Now [first proof], suppose that $\frac{1}{2} \in I$. Since $\frac{1}{2} > 0$, it follows, by Theorem 111, that $\frac{1}{2} \in I^+$. Hence, by Theorem 107a, $\frac{1}{2} \nless 1$. So, if $\frac{1}{2} \in I$ then $\frac{1}{2} \nless 1$. Since, as previously shown, $\frac{1}{2} < 1$, it follows that $\frac{1}{2} \notin I$.
>
> Alternatively [second proof], by Theorem 109, if $\frac{1}{2} \in I$ then $\exists_m \exists_n\ \frac{1}{2} = m - n$. Suppose, now, that $\frac{1}{2} = p - q$--that is, that $p = q + \frac{1}{2}$. Since $0 < \frac{1}{2} < 1$, it follows by the atpi and the cpa that $q + 0 < q + \frac{1}{2} < q + 1$--that is, that $q < p < q + 1$. Since, by Theorems 106 and 88, this is not the case [see Exercise 2 of Part D on page 7-86], it follows that $\frac{1}{2} \neq p - q$. Consequently, $\forall_m \forall_n\ \frac{1}{2} \neq m - n$--that is, it is not the case that $\exists_m \exists_n\ \frac{1}{2} = m - n$. Hence, by Theorem 109, $\frac{1}{2} \notin I$.

*

On pages 7-94 and 7-95 we have, contrary to our usual practice, used the same letters ['m' and 'n' on page 7-94, 'm', 'n', 'j', and 'k' on page 7-95] both as "apparent variables" [in quantified statements] and as "real (i.e. "genuine") variables" [in open sentences]. [Our usual practice is exemplified in the second proof, just above, for '$\frac{1}{2} \notin I$'. There, 'm' and 'n' are used as apparent variables and 'p' and 'q' as real variables.] There are two reasons for distinguishing between apparent variables and real variables. The first reason is that they serve different purposes. As pointed out in the COMMENTARY for page 2-27 in Unit 2, the principal role of apparent variables is to link quantifiers with the desired argument places in a predicate, while real variables, on the other hand, are place holders. For this reason, in a more formal treatment of logic, we would specify at the outset which symbols were to serve as apparent variables and which as real variables and take care that no symbol was permitted to function in both ways. Rather than make a permanent division of our variables into two classes,

e set I is an example of a set of integers which does not have a least ember--the set of even negative integers is another.

: $k > -12\}$ is a set of integers, some negative, which has a least mber.

*

connection with Theorem 113, note that this theorem specifies a ficient condition for a set of integers to have a least member:

$$S \neq \emptyset \text{ and } S \text{ has a lower bound}$$

dents should readily see that this condition is also necessary--if a S of integers has a least member then $S \neq \emptyset$ and S has a lower bound.

*

e proof of Theorem 113 depends on the result established in Part A the Exploration Exercises. In solving these exercises [particularly ercise 4] students have seen that, for each j, $\{k: k > j\}$ is ordered > in just the same way as I^+ is ordered by >. More technically, the ordered sets are isomorphic with respect to >, and t_j is an iso- rphism of the first on the second. [More properly, t_j is an isomor- sm of the system consisting of the set I and the relation > on itself ich maps $\{k: k > j\}$ on I^+. And the function obtained by restricting domain of t_j to $\{k: k > j\}$ is an isomorphism of the system consisting $\{k: k > j\}$ and > on the system consisting of I^+ and >.]

ere have been examples of isomorphic systems in earlier units. For mple, the positiving function, $^+$, is an isomorphism of the system con- ting of the set of numbers of arithmetic with the operations + and ×, l the relation > [for numbers of arithmetic] on the system consisting the set of nonnegative real numbers with the operations + and ×, and relation > [for nonnegative real numbers]. Also, the function $^-$ is l isomorphism of the system consisting of the set of numbers of arith- tic with + and > on the system consisting of the set of nonpositive l numbers with + and <. Here is a third example with which students e not yet acquainted: The system consisting of the set P with × and > isomorphic to the system consisting of the set of real numbers with + l >. This follows from the fact that the common logarithm function, , has for its domain the set P and for its range the set of all real mbers, and is such that

$$\forall_{x \in P} \forall_{y \in P} \log(xy) = \log x + \log y$$

and $$\forall_{x \in P} \forall_{y \in P} [x > y \Longrightarrow \log x > \log y].$$

[It follows, as in Part H on page 7-41, that log has an inverse and that, in the second of the two generalizations above, '$\Longrightarrow$' can be replaced by '$\Longleftrightarrow$'. Hence, log is an isomorphism of the first system on the second.]

Given two isomorphic systems, any theorem about one system can be translated into a statement about the other which is also a theorem. For example, consider the system consisting of $\{k: k > j\}$ with $>$ and that consisting of I^+ with $>$. These systems are isomorphic. Since the notion of least member involves only the relation $>$ [see page 7-87; $<$ is the converse of $>$, and the notion of converse, as well as those of equality and membership, are logical notions], Theorem 108 is a theorem about the second of the two systems. So, the following statement about the first system is also a theorem:

(*) Each nonempty subset of $\{k: k > j\}$ has a least member.

The fourth and fifth paragraphs on page 7-98 show how (*) can be derived from Theorem 108 by using the fact that t_j is an isomorphism. [The explanation asked for in the fourth paragraph is that since t_j maps $\{k: k > j\}$ on I^+, it maps the subset S of the first set on a subset of the second. An answer to the 'Why?' in the fifth paragraph is: Because the only [extra-logical] notion required to define the notion of lower bound is the notion of order, and, by Exercise 4(c) of Part A, t_j preserves order.]

The same kind of argument can be applied to show that, for any theorem about the system consisting of I^+ with $>$, the "corresponding" statement about the system consisting of $\{k: k > j\}$ with $>$ is also a theorem.

The theorem (*) is not quite Theorem 113. But [see the third paragraph on page 7-98], if S is any set of integers which has a lower bound then there is an integer j such that $S \subseteq \{k: k > j\}$. Theorem 113 is an immediate consequence of this and (*). [The explanation asked for in the third paragraph is as follows: As proved in Exercise 4(a) of Part B of the Exploration Exercises, given a lower bound, a, of S, there is an integer $j < a$. It follows [by the definition of 'lower bound' and a theorem like Theorem 92] that j is less than each member of S.]

restriction, rather than as a step in the proof. Because of this restriction on the test-pattern (1) - (5), the conclusion (6) must have a restricted quantifier which honors this restriction. To justify using the quantifier '$\forall_{k \geq 0}$' one must show that if $j \geq 0$ then $j + 1 \in I^+$. To do so, note that, by Theorem 112, $j \geq 0 \Longrightarrow j + 1 > 0$; by Theorem 111, $j + 1 > 0 \Longrightarrow j + 1 - 0 \in I^+$; by Theorem 43, $j + 1 - 0 = j + 1$. So, $j \geq 0 \Longrightarrow j + 1 \in I^+$.

We now turn to (b). In showing that each conditional sentence of this form is a theorem it is convenient to make use of the theorem:

$$\forall_n \ (n - 1 \in I \text{ and } n - 1 \geq 0)$$

This follows easily from previously proved theorems [including Theorems 110c and 104] and we shall not trouble to prove it here. As to (b):

(1)	$\forall_{k \geq 0} F(k)$	[assumption]*
(2)	$\forall_n \ (n - 1 \in I \text{ and } n - 1 \geq 0)$	[theorem]
(3)	$p - 1 \in I \text{ and } p - 1 \geq 0$	[(2)]
(4)	$F(p - 1)$	[(1), (3)]
(5)	$\forall_n F(n - 1)$	[(1) - (5)]
(6)	$\forall_{k \geq 0} F(k) \Longrightarrow \forall_n F(n - 1)$	[(5); *(1)]

In explanation: Step (3) is the auxiliary premiss which is required by the rule for restricted universal instantiation in order to infer (4) from (1).

As has been pointed out in the COMMENTARY for Part D on page 7-68, it is sometimes more convenient to use a slightly different induction principle which is equivalent to the one above. This principle is:

$$\forall_S [(0 \in S \text{ and } \forall_n [n - 1 \in S \Rightarrow n \in S]) \Rightarrow \forall_n n - 1 \in S]$$

The equivalence of the two principles follows almost at once by virtue of the substitution rule for biconditional sentences once one establishes the theorems:

$$\forall_{k \geq 0} [k \in S \Rightarrow k + 1 \in S] \Longleftrightarrow \forall_n [n - 1 \in S \Rightarrow (n - 1) + 1 \in S]$$

$$\forall_{k \geq 0} k \in S \Longleftrightarrow \forall_n n - 1 \in S$$

[One also needs the theorem '$\forall_n (n - 1) + 1 = n$'--but this is a consequence of Theorem 32.] Since both the needed biconditionals are of the form:

$$\forall_{k \geq 0} F(k) \Longleftrightarrow \forall_n F(n - 1),$$

it is sufficient [and a saving of time] to show that each biconditional of this form is a theorem. And to do this it is sufficient to show that each conditional sentence of either of the forms:

(a) $\forall_n F(n - 1) \Rightarrow \forall_{k \geq 0} F(k)$ (b) $\forall_{k \geq 0} F(k) \Rightarrow \forall_n F(n - 1)$

is a theorem. We begin with (a).

(1)	$\forall_n F(n - 1)$		[assumption]*
(2)	$F(j + 1 - 1)$	$[j + 1 \in I^+]$	[(1)]
(3)	$\forall_x \forall_y x + y - y = x$		[Theorem 30]
(4)	$j + 1 - 1 = j$		[(3)]
(5)	$F(j)$		[(2), (4)]
(6)	$\forall_{k \geq 0} F(k)$		[(1) - (5), Ths. 112, 111, 43]
(7)	$\forall_n F(n - 1) \Rightarrow \forall_{k \geq 0} F(k)$		[(6); *(1)]

In explanation: The restriction on step (2) is the auxiliary premiss which is required by the rule for restricted universal instantiation [see COMMENTARY for page 7-36] in order to infer (2) from (1). By the rule (R_1) [same COMMENTARY], this premiss may, as here, be entered as a

Continuing the discussion of the COMMENTARY for page 7-98, we take up another illustration of the fact that isomorphic systems have similar theorems. For notational convenience, we deal with the system consisting of $\{k: k \geq j\}$ with $>$, rather than as before, with the system consisting of $\{k: k > j\}$ with $>$. Like the latter, the former is isomorphic to the system consisting of I^+ with $>$. Our aim is to use an isomorphism argument to establish a principle of mathematical induction [Theorem 114] for the former system.

In order to do so, we note, first, that [by Theorem 112] $\{k: k \geq j\} = \{k: k > j - 1\}$ and [by Theorem 111] $I^+ = \{k: k > 0\}$. Using the latter, (I^+_3) becomes:

$$(*) \qquad \forall_S [(1 \in S \text{ and } \forall_{k > 0} [k \in S \Longrightarrow k + 1 \in S]) \Longrightarrow \forall_{k > 0}\, k \in S]$$

Since the isomorphism between the two systems does not map 1 into 1 and does not preserve addition, (*) is not yet in a form which can be translated into a theorem about the system consisting of $\{k: k > j - 1\}$ and $>$.

To put (*) in suitable form, we note, next, that for each k,

$$(**) \qquad k + 1 = \text{the least member of } \{i: i > k\}.$$

In fact, by Theorem 90, $k + 1 \in \{i: i > k\}$ and, by Theorem 112, $k + 1$ is a lower bound of $\{i: i > k\}$. [By Theorem 112, $(i - 1) + 1 > k \Longrightarrow i - 1 \geq k$; so, if $i > k$ then $i \geq k + 1$.] So, in '$1 \in S$', '1' may be replaced by 'the least member of $\{i: i > 0\}$' and, in '$k + 1 \in S$', '$k + 1$' may be replaced by 'the least member of $\{i: i > k\}$'. Furthermore, the restriction '$k > 0$' in the two quantifiers may be replaced by '$k \in \{i: i > 0\}$'. [At this point, it will be worth your while to take a large sheet of paper and write out the statement obtained by making these four replacements in (*). Compare it with the display just above the middle of page 7-99.]

Since the only extra-logical notion involved in the notion of least member is that of $>$, and since the ordered sets $\{i: i > 0\}$ and $\{i: i > j - 1\}$ are isomorphic, the resulting theorem implies the sentence obtained by replacing '$\{i: i > 0\}$' by '$\{i: i > j - 1\}$'. [Write it down.]

Now, invoking (**), one may replace 'the least member of $\{i: i > j - 1\}$' by 'j'; '$k \in \{i: i > j - 1\}$' by '$k \geq j$'; and 'the least member of $\{i: i > k\}$' by '$k + 1$'. The result [when preceded by '$\forall_j$'] is Theorem 114.

*

One of the more useful instances of Theorem 114 is:

$$\forall_S [(0 \in S \text{ and } \forall_{k \geq 0} [k \in S \Longrightarrow k + 1 \in S]) \Longrightarrow \forall_{k \geq 0}\, k \in S]$$

Since, as shown in Part B of the Exploration Exercises, the function r is an isomorphism of the system consisting of I with > on the system consisting of I with < [and, since r is its own inverse, vice versa], each theorem about integers in which the only extra-logical terms are 'I' and '>' [or '<'] is transformed into a theorem if '>' is replaced by '<' [and '<' by '>']. If Theorem 113 is rewritten by using the definitions on page 7-87 of 'lower bound' and 'least member' in such a way that the only extra-logical terms remaining are 'I' and '<' then replacing '<' by '>' results in a statement equivalent to Theorem 115.

*

To obtain Theorem 116 from Theorem 114, recall that, for each k,

$$k + 1 = \text{the least member of } \{i: i > k\}.$$

Replacing '$k + 1$' in Theorem 114 by the right side of the equation above and then replacing 'least' by 'greatest' and '>' by '<' results in a statement which, since $k - 1$ = the greatest member of $\{i: i < k\}$, is equivalent to Theorem 116.

*

From Theorems 114 and 116 one obtains as instances:

$$(0 \in S \text{ and } \forall_{k \geq 0} [k \in S \Longrightarrow k + 1 \in S]) \Longrightarrow \forall_{k \geq 0}\, k \in S$$

and:

$$(0 \in S \text{ and } \forall_{k \leq 0} [k \in S \Longrightarrow k - 1 \in S]) \Longrightarrow \forall_{k \leq 0}\, k \in S$$

Assuming that $0 \in S$ and that $\forall_k [k \in S \Longrightarrow (k + 1 \in S \text{ and } k - 1 \in S)]$, it follows easily that $\forall_{k \geq 0}\, k \in S$ and $\forall_{k \leq 0}\, k \in S$. Since, for each k, $k \geq 0$ or $k \leq 0$, it follows [by the rule of the dilemma] that $\forall_k\, k \in S$. Consequently, Theorem 117.

*

Answers for Exercises.

1. 0; $n - 1$

2. $\dfrac{n(n - 3)}{2}$

*

[In answering Exercise 2, students may find it enlightening to compare the answer for Exercise 1 with the recursive definition for T given on page 7-61. From the latter it follows that

$$\forall_{n \geq 3}\ T_{n-1} = T_{n-2} + (n - 1).$$

Comparing this with:

$$\forall_{n \geq 3}\ D_{n+1} = D_n + (n - 1)$$

suggests comparing D_n and T_{n-2}. Since $D_3 = 0$ and $T_{3-2} = 1$, it seems likely [and, in fact, is easily proved] that, for each n, $D_n = T_{n-2} - 1$. Now, use of (2) on page 7-63 yields the correct answer for Exercise 2.]

*

3. (i) By the recursive definition, $D_3 = 0 = \frac{3(3-3)}{2}$.

   (ii) Suppose that $D_q = \frac{q(q-3)}{2}$ [for an integer $q \geq 3$]. By the recursive definition, it follows that $D_{q+1} = D_q + (q - 1) = \frac{q(q-3)}{2} + (q - 1) = \frac{(q+1)[(q+1)-3]}{2}$. Consequently, $\forall_{n \geq 3}\ [D_n = \frac{n(n-3)}{2} \Rightarrow D_{n+1} = \frac{(n+1)[(n+1)-3]}{2}]$.

   (iii) From (i) and (ii) it follows, by Theorem 114 [j = 3], that $\forall_{n \geq 3}\ D_n = \frac{n(n-3)}{2}$.

*

In a more complete form, part (iii) of the answer for Exercise 3 would bear a closer analogy with the form given at the foot of page 7-53. Expanded into a column it would, as shown on page 7-53, consist of six steps:

$$3 \in \{m \geq 3:\ D_m = \frac{m(m-3)}{2}\}$$

$$\forall_{n \geq 3}[n \in \{m \geq 3:\ D_m = \frac{m(m-3)}{2}\} \Rightarrow n + 1 \in \{m \geq 3:\ D_m = \frac{m(m-3)}{2}\}]$$

[Theorem 114]

[instance of Theorem 114: $j = 3$; $S = \{m \geq 3:\ D_m = \frac{m(m-3)}{2}\}$]

$$\forall_{n \geq 3}\ n \in \{m \geq 3:\ D_m = \frac{m(m-3)}{2}\}$$

$$\forall_{n \geq 3}\ D_n = \frac{n(n-3)}{2}$$

Answers for Miscellaneous Exercises.

[<u>easy</u>: Part B; <u>medium</u>: Part A]

Answers for Part A [This is important preparation for Unit 8.].

1. (a) $\frac{1}{6}$; $\frac{1}{12}$; $\frac{1}{20}$; $\frac{1}{30}$

   (b) $\forall_n\ f_n = \frac{1}{n(n+1)}$;

   (i) $f_1 = \frac{1}{2} = \frac{1}{1(1+1)}$

   (ii) Suppose that $f_p = \frac{1}{p(p+1)}$. Since $f_{p+1} = f_p \cdot \frac{p}{p+2}$, it follows that $f_{p+1} = \frac{1}{(p+1)(p+2)}$. Consequently, $\forall_n\ [f_n = \frac{1}{n(n+1)} \Rightarrow f_{n+1} = \frac{1}{(n+1)[(n+1)+1]}]$.

   (iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\ f_n = \frac{1}{n(n+1)}$.

2. $\frac{n}{n+1}$

3. $2(n+1)$

4. (a) $n+1$ (b) $n+2$ (c) $\frac{n+3}{n+4}$ (d) $\frac{1}{(n+4)(n+5)}$

*

Answers for Part B.

1. (a) $x^2 - x + 1$ (b) $x^2 + x + 1$

2. -12

3. 12.6

4. 3

*

For each x, x is an upper bound of $\{k: k \leq x\}$. [line 5 on page 7-102]

*

(1) 4	(2) 2	(3) 4	(4) 3	(5) 8
(6) −2	(7) 1	(8) 0	(9) −1	(10) 2
(11) 7	(12) 0	(13) −6	(14) 4	(15) −6

(16) 0 (17) between the '3' and the '4' (18) same as (17)

(19) between the '1' and the '2'; between the '1' and the '2'; between the '2' and the '3'; between the '−4' and the '−5'

(f) (i) [Ask students: Why '$k > 0$'? (Answer: To use the mtpi.)]

(g) (i); Since $[\![c]\!] \in I$, it follows, by Theorem 110b, for any integer k, that $[\![c]\!] + k \in I$. So, by Theorem 118a, $[\![c]\!] + k \leq [\![c + k]\!]$ if and only if $[\![c]\!] + k \leq c + k$. By the atpi, it follows that $[\![c]\!] + k \leq c + k$ if and only if $[\![c]\!] \leq c$. Since, by part (a), $[\![c]\!] \leq c$, it follows that $[\![c]\!] + k \leq [\![c + k]\!]$. Consequently, (i).

(h) Since $[\![a]\!] \in I$ and $[\![b]\!] \in I$, it follows, by Theorem 110b, that $[\![a]\!] + [\![b]\!] \in I$. So, by Theorem 118a, $[\![a]\!] + [\![b]\!] \leq [\![a + b]\!]$ if and only if $[\![a]\!] + [\![b]\!] \leq a + b$. By part (a) and Theorem 91, this is the case. So, $[\![a]\!] + [\![b]\!] \leq [\![a + b]\!]$. Consequently, $\forall_x \forall_y\, [\![x]\!] + [\![y]\!] \leq [\![x + y]\!]$.

*

As an extension of Exercise 3(h) of Part A, and in anticipation of Part D on page 7-106, you might ask students to investigate the generalization:

(*) $$\forall_x \forall_y\, [\![x]\!] + [\![y]\!] = [\![x + y]\!]$$

For example, ask them to complete:

(a) $[\![2.43]\!] + [\![3.25]\!] =$ (b) $[\![2.43 + 3.25]\!] =$

(c) $[\![1.27]\!] + [\![-3.9]\!] =$ (d) $[\![1.27 - 3.9]\!] =$

(e) $[\![4.68]\!] + [\![3.51]\!] =$ (f) $[\![4.68 + 3.51]\!] =$

[Answers: (a) 5, (b) 5, (c) −3, (d) −3, (e) 7, (f) 8]

Students should guess that, although (*) is <u>not</u> a theorem, the following is:

(**) $$\forall_x \forall_y\, [\![x + y]\!] \leq [\![x]\!] + [\![y]\!] + 1$$

In explaining the situation, students will probably discover the <u>fractional part</u> function introduced on page 7-107. [Explanation: $[\![a+b]\!] = [\![a]\!] + [\![b]\!]$ if $\{\!\{a\}\!\} + \{\!\{b\}\!\} < 1$, but $[\![a + b]\!] = [\![a]\!] + [\![b]\!] + 1$ if $\{\!\{a\}\!\} + \{\!\{b\}\!\} \geq 1$.] The theorem (**) can be proved by using Theorem 118c on page 7-105 [see Part D on page 7-106].

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(g) (i); Since $[\![c]\!] \in I$, it follows, by Theorem 110b, for any integer k, that $[\![c]\!] + k \in I$. So, by Theorem 118a, $[\![c]\!] + k \leq [\![c + k]\!]$ if and only if $[\![c]\!] + k \leq c + k$. By the atpi, it follows that $[\![c]\!] + k \leq c + k$ if and only if $[\![c]\!] \leq c$. Since, by part (a), $[\![c]\!] \leq c$, it follows that $[\![c]\!] + k \leq [\![c + k]\!]$. Consequently, (i).

(h) Since $[\![a]\!] \in I$ and $[\![b]\!] \in I$, it follows, by Theorem 110b, that $[\![a]\!] + [\![b]\!] \in I$. So, by Theorem 118a, $[\![a]\!] + [\![b]\!] \leq [\![a + b]\!]$ if and only if $[\![a]\!] + [\![b]\!] \leq a + b$. By part (a) and Theorem 91, this is the case. So, $[\![a]\!] + [\![b]\!] \leq [\![a + b]\!]$. Consequently, $\forall_x \forall_y\ [\![x]\!] + [\![y]\!] \leq [\![x + y]\!]$.

*

As an extension of Exercise 3(h) of Part A, and in anticipation of Part D on page 7-106, you might ask students to investigate the generalization:

(*) $$\forall_x \forall_y\ [\![x]\!] + [\![y]\!] = [\![x + y]\!]$$

For example, ask them to complete:

(a) $[\![2.43]\!] + [\![3.25]\!] =$ (b) $[\![2.43 + 3.25]\!] =$

(c) $[\![1.27]\!] + [\![-3.9]\!] =$ (d) $[\![1.27 - 3.9]\!] =$

(e) $[\![4.68]\!] + [\![3.51]\!] =$ (f) $[\![4.68 + 3.51]\!] =$

[Answers: (a) 5, (b) 5, (c) -3, (d) -3, (e) 7, (f) 8]

Students should guess that, although (*) is <u>not</u> a theorem, the following is:

(**) $$\forall_x \forall_y\ [\![x + y]\!] \leq [\![x]\!] + [\![y]\!] + 1$$

In explaining the situation, students will probably discover the <u>fractional part</u> function introduced on page 7-107. [Explanation: $[\![a + b]\!] = [\![a]\!] + [\![b]\!]$ if $\{\!\{a\}\!\} + \{\!\{b\}\!\} < 1$, but $[\![a + b]\!] = [\![a]\!] + [\![b]\!] + 1$ if $\{\!\{a\}\!\} + \{\!\{b\}\!\} \geq 1$.] The theorem (**) can be proved by using Theorem 118c on page 7-105 [see Part D on page 7-106].

Proof of Theorem 118a: Since [for any real number c] $[\![c]\!]$ = the greatest member of $\{k: k \leq c\}$, it follows that $[\![c]\!] \leq c$ and $\forall_k\, [k \leq c \Longrightarrow k \leq [\![c]\!]]$. Using the former, it follows that $\forall_k\, [k \leq [\![c]\!] \Longrightarrow k \leq c]$. Consequently, Theorem 118a.

*

Notice that from '$\forall_x\, [\![x]\!] \in I$' and '$\forall_x \forall_k\, [k \leq [\![x]\!] \Longrightarrow k \leq x]$' follows '$\forall_x\, [\![x]\!] \leq x$'. Notice, also, that '$\forall_x \forall_k\, [k \leq x \Longrightarrow k \leq [\![x]\!]]$' says that, for each x, $[\![x]\!]$ is an upper bound of $\{k: k \leq x\}$. So, Theorem 118a and '$\forall_x\, [\![x]\!] \in I$' are, together, equivalent to '$\forall_x\, [\![x]\!]$ = the greatest member of $\{k: k \leq x\}$'.

*

Answers for Part A.

1. 4; 4.3

2. (f), (j)

3. (a) By Theorem 118a, since $[\![c]\!] \in I$, it follows that $[\![c]\!] \leq [\![c]\!]$ if and only if $[\![c]\!] \leq c$. So, since $[\![c]\!] \leq [\![c]\!]$, it follows that $[\![c]\!] \leq c$. Consequently, $\forall_x\, [\![x]\!] \leq x$.

   (b) By part (a), $[\![c]\!] \leq c$. So, since $2 > 0$, it follows from the mtpi that $2[\![c]\!] \leq 2c$. Consequently, $\forall_x\, 2[\![x]\!] \leq 2x$.

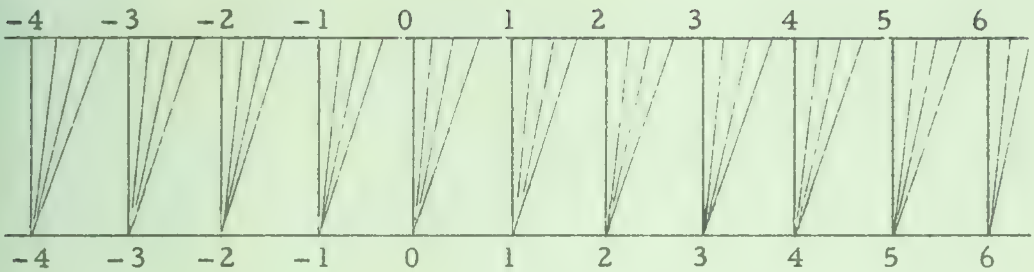
   (c) Since $[\![c]\!] \in I$ and $2 \in I$ it follows, from Theorem 110d, that $2[\![c]\!] \in I$. So, by Theorem 118a, $2[\![c]\!] \leq [\![2c]\!]$ if and only if $2[\![c]\!] \leq 2c$. Since, by part (b), $2[\![c]\!] \leq 2c$, it follows that $2[\![c]\!] \leq [\![2c]\!]$. Consequently, $\forall_x\, 2[\![x]\!] \leq [\![2x]\!]$.

   (d) [Similar to (b)]

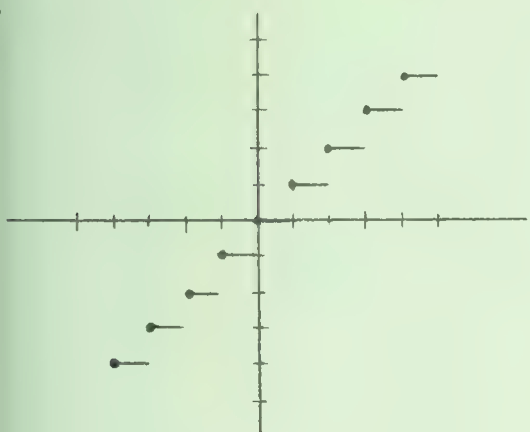
   (e) 1 is a counter-example. $[\frac{1}{2}[\![1]\!] = \frac{1}{2},\ [\![\frac{1}{2} \cdot 1]\!] = 0,\ \frac{1}{2} \not\leq 0]$
   [Students should see that an attempt to use (d) to prove (e) as (b) was used to prove (c) breaks down because it is not the case that $\forall_x\, \frac{1}{2}[\![x]\!] \in I$. So, '$\frac{1}{2}[\![c]\!]$' cannot be used for 'k' in Theorem 118a.]

Answers for Part B.

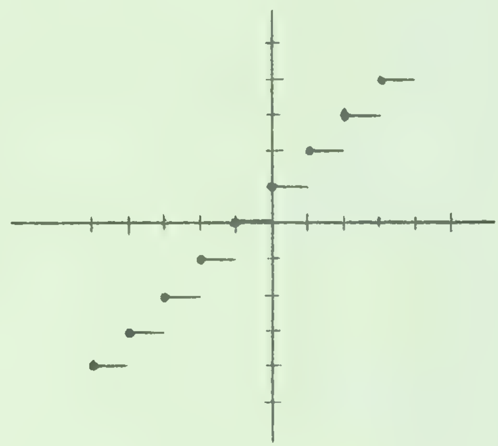
1.



2.



3.



*

Explanation concerning (*) on page 7-104: By Theorem 118a, $k \leq \llbracket c \rrbracket$ if and only if $k \leq c$. So [by biconditional substitution into the logically valid sentence 'not ($k \leq \llbracket c \rrbracket$) if and only if not ($k \leq \llbracket c \rrbracket$)'], it follows that $k \not\leq \llbracket c \rrbracket$ if and only if $k \not\leq c$. From this and Theorem 88 [again, by the substitution rule for biconditional sentences], (*) follows.

In deriving (*) from Theorems 118a and 88, we have proved:

$$\forall_x \forall_k \, [k > \llbracket x \rrbracket \iff k > x]$$

This is Theorem 118b.

Explanation asked for in line 2 on page 7-105: (*), on page 7-104, holds for any integer k--that is, on page 104 we have actually derived '$\forall_k\ [k > [\![c]\!] \Longleftrightarrow k > c]$'. So, since $k + 1 \in I$, $k + 1 > [\![c]\!] \Longleftrightarrow k+1 > c$.

*

Just as Theorem 118b [see end of COMMENTARY for page 7-104] was derived from Theorems 118a and 88, so, from Theorems 118c and 88 one can derive Theorem 118d:

$$\forall_x \forall_k\ [k < [\![x]\!] \Longleftrightarrow k + 1 \leq x]$$

*

The answer to the 'Why?' in the test-pattern displayed on page 7-105 is, of course, 'atpi'.

*

Notice that, from '$\forall_x\ [\![x]\!] \in I$' and the only-if-part of Theorem 118e, follows:

$$\forall_x\ [\ [\![x]\!] \leq x < [\![x]\!] + 1]$$

*

Answer for Part C.

By Theorem 118e,

$$[\![c]\!] + j = [\![c + j]\!] \text{ if and only if } [\![c]\!] + j \leq c + j < [\![c]\!] + j + 1.$$

By the atpi, this is the case if and only if $[\![c]\!] \leq c < [\![c]\!] + 1$. But, since $[\![c]\!] \in I$ [and $[\![c]\!] = [\![c]\!]$], it follows, by Theorem 118e, that this is the case. Hence, $[\![c]\!] + j = [\![c + j]\!]$. Consequently, $\forall_x \forall_j\ [\![x + j]\!] = [\![x]\!] + j$.

An approach to Part D is suggested in the COMMENTARY, in connection with Exercise 3(h) of Part A on page 7-103.

*

Answer for Part D.

$\forall_x \forall_y [\![x + y]\!] \leq [\![x]\!] + [\![y]\!] + 1;$

By Theorem 118c, since $[\![a]\!] + [\![b]\!] + 1 \in I$, it follows that

$$[\![a + b]\!] \leq [\![a]\!] + [\![b]\!] + 1 \text{ if and only if } ([\![a]\!] + [\![b]\!] + 1) + 1 > a + b.$$

Since, by Theorem 118c, $a < [\![a]\!] + 1$ and $b < [\![b]\!] + 1$, it follows with the use of Theorem 91 that this is the case. Hence, $[\![a + b]\!] \leq [\![a]\!] + [\![b]\!] + 1$. Consequently, $\forall_x \forall_y \ldots$.

*

Answers for Part E.

1. 5; Since, for $b > 0$, $b \neq 0$, it follows from Theorem 118e that $[\![\frac{a}{b}]\!] \leq \frac{a}{b} < [\![\frac{a}{b}]\!] + 1$. Hence, by the mtpi and the pq, for $b > 0$, $[\![\frac{a}{b}]\!]b \leq a < ([\![\frac{a}{b}]\!] + 1)b$. So, there is an integer k such that $kb \leq a < (k + 1)b$. Consequently, $\forall_x \forall_{y > 0} \exists_k\, ky \leq x < (k + 1)y$.

2. (a) By the theorem of Exercise 1, for $b > 0$, there is an integer, say k, such that $kb \leq a < (k + 1)b$. It follows, using the atpi, that $0 \leq a - kb < b$. Consequently, $\forall_x \forall_{y > 0} \exists_k\, 0 \leq x - ky < y$.

   (b) By algebra, $a = kb + (a - kb)$. By the theorem of part (a), for $b > 0$, there is an integer, say k, such that $0 \leq a - kb < b$. Hence, $\exists_z\, (a = kb + z \text{ and } 0 \leq z < b)$. Consequently, ...

*

Theorem 120 says that, for $b > 0$, one can "divide" a by b to obtain an integral <u>partial quotient</u> and a nonnegative <u>remainder</u> which is less than b. The division-with-remainder algorithm on page 7-12 gives a procedure for computing these numbers when given decimal representations of a and b.

*

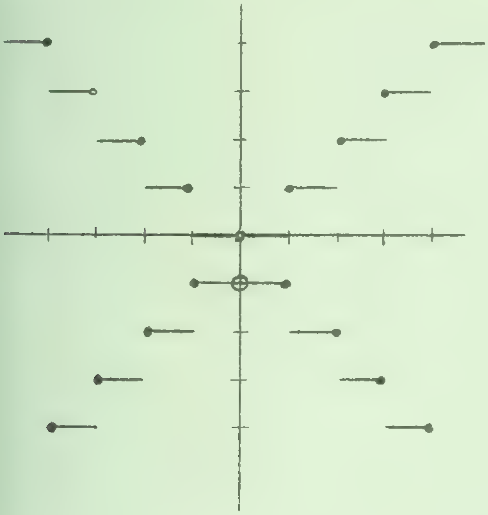
3. By the cofinality principle, there is [for $b \neq 0$] a positive integer, say p, such that $p > \frac{a}{b}$. Since, for $b > 0$, $b \neq 0$, it follows by the mtpi and the pq that, for $b > 0$, $pb > a$. Consequently, $\forall_x \forall_{y > 0} \exists_n\, ny > x$.

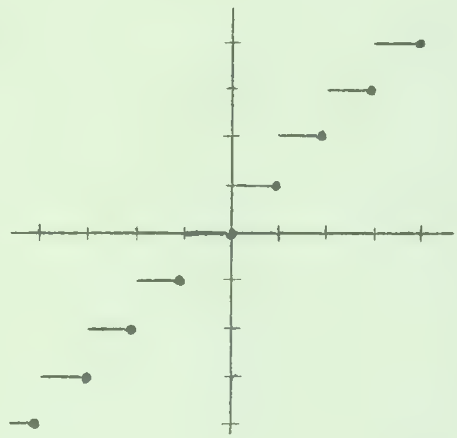
*

Theorem 121 is the Archimedean principle.

*

Answers for Part F.

1. 

2. 

[The graph of '$y = [\![-x]\!]$' is the reflection of that of '$y = [\![x]\!]$' in the graph of the y-axis. That of '$y = -[\![-x]\!]$' is the reflection of the graph of '$y = [\![-x]\!]$' in the graph of the x-axis. It is also the reflection of the graph of '$y = [\![x]\!]$' through the graph of the origin.]

3. least; $k \geq$ [The function f such that $f(x) = -[\![-x]\!]$ is called the least integer function.]

4. $c = 9 - 5\,[\![1 - w]\!]$

Part ☆J and pages 7-108 through 7-111 are optional. Their purpose is to bring out an algorithm for finding, for any integer $b > 1$, the base-b representation of any given positive integer; and to prove Theorems 123 and 124. The latter is used in Unit 8 to prove that each integer has, for each $b > 1$, a base-b representation.

*

Answers for Part ☆J.

1. 1 2. 2 3. 6 4. 5 5. 7 6. 0 7. 0

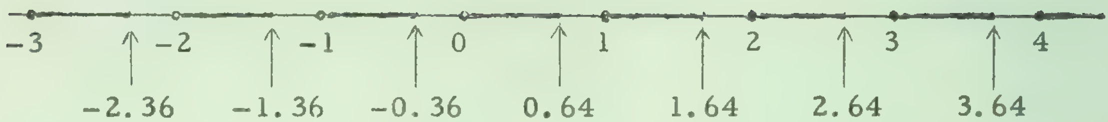
8. 55037 9. (a) 2 (b) 0

10. 5; 4; 5; 3; 2; 2; $\forall_{k > 6} \left[\!\left[\left\{ \frac{75621}{8^k} \right\} 8 \right]\!\right] = 0$

11. $4 \cdot 8 + 5 \cdot 8^2 + 3 \cdot 8^3 + 2 \cdot 8^4 + 2 \cdot 8^5$

12. (a) 2 (b) 10 (c) 0 (d) 0

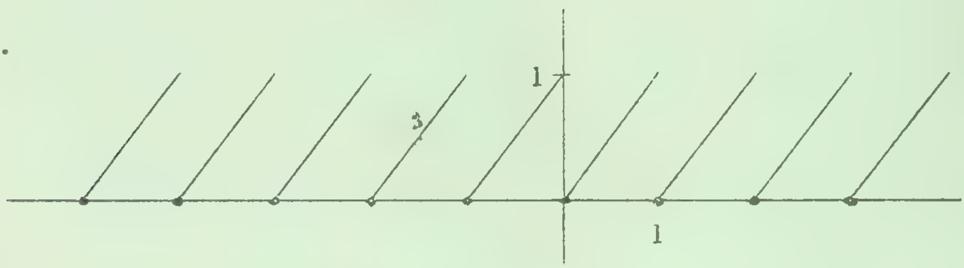
Answer for Part G.



*

Answers for Part H.

1.



2. $\forall_x \forall_j \ \{\!\{x + j\}\!\} = \{\!\{x\}\!\}$;

$\{\!\{a + j\}\!\} = (a + j) - [\![a + j]\!] = (a + j) - ([\![a]\!] + j) = a - [\![a]\!] = \{\!\{a\}\!\}$

Consequently, $\forall_x \forall_j \ldots$.

*

By the theorem of Exercise 2, the fractional part function is <u>periodic</u>. [A function f is periodic if there is a $y \neq 0$ such that, for each $x \in \mathscr{D}_f$, $x + y \in \mathscr{D}_f$ and $f(x + y) = f(x)$. Such a number y is called a <u>period</u> of f.] More specifically, the theorem of Exercise 2 says that each nonzero integer is a period of the fractional part function.

*

Answers for Part I.

1. $\{x: \{\!\{x\}\!\} < 0.64\}$

2. $\{x: \{\!\{x\}\!\} \geq 0.64\}$

*

Correction. On page 7-112, line 5 should begin:

$$(c)\ \forall_n\ \frac{n(n+1)(n+2)(n+3)}{4} +$$

and on page 7-113, line 2 should end:

---area-measure of $\triangle ABC$?

Answers for Miscellaneous Exercises.

[easy: A 1-3, B 1-9, 11-14, 16-24; medium: B 15; hard: B 10]

Answers for Part A [This is important preparation for Unit 8.].

1. (a) $n + 2$ (b) $n + 2$; $n + 3$ (c) $(n + 1)(n + 2)(n + 3)(n + 4)$

2. $n + 2$; $2n + 3$

3. $p + 1$; $2p + 1$; $2p + 3$

[The proofs given for the generalization in Part A should be kept very simple. For example, for Exercise 1 it is enough to write:

$$\frac{p(p+1)}{2} + (p+1) = \frac{p(p+1) + 2(p+1)}{2} = \frac{(p+1)(p+2)}{2}$$]

*

Answers for Part B.

1. (a) overestimate (except for rectangles) [If a and c are measures of parallel sides then the area-measure of the quadrilateral is $\frac{a+c}{2}\cdot h$. Since $h \leq b$ and $h \leq d$, $h \leq \frac{b+d}{2}$.]

   (b) rectangles

2. about 3450000

3. (C)

4. 10

5. (a) $-12a^6b^7$ (b) $-21p^6q^{13}$ (c) $80x^3y^5z^4$ (d) $64r^2s^9t^6u$

6. $1800/x$ [perhaps: $[\![1800/x]\!]$]

7. about 23

8. 0.625

9. (a) 1 (b) $\frac{67}{60}$ (c) 4 (d) $\frac{x+y}{2xy}$

10. $\left(\frac{284.8869792}{2.640834818}, \frac{36.990115510368}{2.640834818}\right)$ [approx. (107.8776215, 14.00697811)]

11. $60/k$ [perhaps: $[\![60/k]\!]$]

12. 15 [all]

13. 50 miles per hour

14. $\ell = \dfrac{2S - na}{n}$

15. 23 hours, 59 minutes, and 30 seconds

16. (a) $\dfrac{2x}{(x - 4)(x - 3)}$ (b) $\dfrac{2}{(a + 1)(a + 2)}$

17. $r = \dfrac{pq}{p + q}$

18. $\dfrac{22x}{15y}$

19. overestimate

20. $b = \dfrac{cd}{a}$

21. 2

22. (C)

23. The measure of the hypotenuse of a right triangle is less than the sum of the measures of its legs.

24. 50

Answers for Part A.

1. t 2. t 3. f 4. f 5. f 6. t 7. t 8. t 9. t

[In discussing Exercise 9, bring out the point that, since '$8|80$' is a sentence, '$(8|80) + 888$' would be nonsense. So, there is no need for parentheses in '$8|(80 + 888)$'. Also, some student should suggest that the sentence is true because $8|80$ and $8|888$ (see Exercise 3 at foot of page 7-115).]

10. t 11. f 12. f [Note that, since $3|63$ but $3\nmid 59$, $3\nmid 63 + 59$ (see Exercise 4 at top of page 7-116).]

13. t [Note that, since $7|14$, it follows that $7|14 \cdot 8273591$ (see Exercise 5 on page 7-116). Similar remarks apply to Exercises 14, 15 and 16.]

14. t 15. t 16. t

17. t [Note that although 6 does not divide either 103, 104, or 105, it does divide their product.]

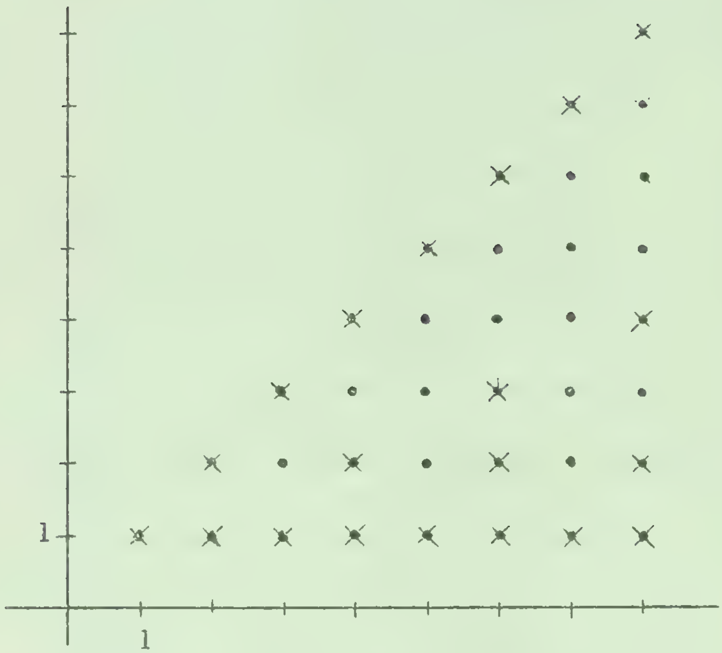
*

Answers for Part B.

1. Suppose that $m|n$ and $n|p$. Then, by definition, there are positive integers, say q_1 and q_2, such that $n = mq_1$ and $p = nq_2$. So, $p = (mq_1)q_2 = m(q_1q_2)$. Since, by Theorem 103, $q_1q_2 \in I^+$, it follows, by definition, that $m|p$. Hence, if $m|n$ and $n|p$ then $m|p$. Consequently, $\forall_m \forall_n \forall_p\ [(m|n \text{ and } n|p) \Longrightarrow m|p]$.

2. Suppose that $m|n$ and $n|m$. It follows, from the Sample, that $m \leq n$ and $n \leq m$. So, by Theorem 93, $m = n$. Consequently, $\forall_m \forall_n\ [(m|n \text{ and } n|m) \Longrightarrow m = n]$.

3. Suppose that $m|n$ and $m|p$. Then, by definition, there are positive integers, say q_1 and q_2, such that $n = mq_1$ and $p = mq_2$. So, $n + p = mq_1 + mq_2 = m(q_1 + q_2)$. Since, by Theorem 102, $q_1 + q_2 \in I^+$, it follows, by definition, that $m|n + p$. Consequently, $\forall_m \forall_n \forall_p\ [(m|n \text{ and } m|p) \Longrightarrow m|n + p]$.



Building up a graph such as this on the board may help drive home Theorem 125 [the crosses in the bottom row come from '$\forall_n\ 1|n$' and those on the diagonal from '$\forall_n\ n|n$'] and discover Theorem 126a of the Sample.

*

Be sure students understand that '|' stands for 'divides', and that they don't confuse it with the fraction slash, '/'. One way to check on this is to do the exercises of Part A on page 7-115 in class.

*

In more advanced courses the divisibility relation is usually defined so that its domain and range are I, rather than, as here, I^+. [To do so, all one need do is replace 'm', 'n', and 'p' in line 4 on page 7-115 by 'i', 'j', and 'k', respectively.] With this definition, $\forall_j\ [0|j \Longleftrightarrow j = 0]$ and, in particular, $0|0$. On page 7-129, we have found it convenient to extend the definition so that the domain of | is I, but the range remains I^+. So, throughout this unit, '$0|2$', for example, is meaningless.

As noted on page 7-116, the divisibility relation [mentioned in Unit 5--see page 5-6] has many properties in common with the less-than-or-equal-to relation. This suggests the following introduction, which, to be effective, must precede students' reading of page 7-115.

Teacher: I am thinking of a certain relation among the positive integers. Call it '☆'. Here are some of its properties [list on board]:

$$\forall_n\ 1 \star n$$

$$\forall_n\ n \star n$$

$$\forall_m \forall_n \forall_p\ [(m \star n \text{ and } n \star p) \Longrightarrow m \star p]$$

$$\forall_m \forall_n\ [(m \star n \text{ and } n \star m) \Longrightarrow m = n]$$

$$\forall_m \forall_n \forall_p\ [m \star n \Longleftrightarrow mp \star np]$$

Who can guess what it is?

Student: Less-than-or-equal-to?

Teacher: It could be. [Check this.] But, I wasn't thinking of $\leq$. Here are some statements:

True	False
$5 \star 10$	$5 \star 11$
$3 \star 6$	$3 \star 7$
$10 \star 150$	$10 \star 99$

[Continue listing until some student describes the relation as the divisibility relation.]

*

Recall [Unit 5] that a relation is a set of ordered pairs--$(a, b) \in R$ if and only if $b \mathrel{R} a$, and $R = \{(x, y) \in \mathcal{D}_R \times \mathcal{R}_R : y \mathrel{R} x\}$. The theorem of the Sample in Part B says that the relation $|$ is a subset of the relation $\leq$. In the graph on the following page, members of $\leq$ are shown by heavy dots and members of $|$ are shown by crosses.

Suppose that $m|n$ and $m|n + p$. Then, by definition, there are integers, say q_1 and q_2, such that $n = mq_1$ and $n + p = mq_2$. Since $p = (n + p) - n$, it follows that $p = mq_2 - mq_1 = m(q_2 - q_1)$. Since, by Theorem 101, $p > 0$ and $m > 0$, it follows, by Theorems 96a and 84, that $q_2 > q_1$. So, by Theorem 105, $q_2 - q_1 \in I^+$. Hence, by definition, $m|p$. Consequently, $\forall_m \forall_n \forall_p$ $[(m|n \text{ and } m|n + p) \Rightarrow m|p]$.

Suppose that $m|n$. By definition, there is a positive integer, say q, such that $n = mq$. So, $np = (mq)p = (mp)q$, and, by Theorem 103 and definition, $mp|np$. Consequently, $\forall_m \forall_n \forall_p$ $[m|n \Rightarrow mp|np]$.

(i) Since $1(1 + 1) = 2 \cdot 1$, and since $1 \in I^+$, it follows, by definition, that $2|1(1 + 1)$.

(ii) Suppose that $2|p(p + 1)$.
Since $(p + 1)(p + 2) = p(p + 1) + 2(p + 1)$ and $p + 1 \in I^+$, it follows, by definition and Theorem 126d, that $2|(p + 1)(p + 2)$.
Consequently, $\forall_n$ $[2|n(n + 1) \Rightarrow 2|(n + 1)([n + 1] + 1)]$.

[Note that the theorem of Exercise 6 says that the product of two consecutive positive integers is even; that of Exercise 7 says that the product of three consecutive positive integers is divisible by 6.]

(i) Since $1 \cdot 2 \cdot 3 = 6 \cdot 1$, and since $1 \in I^+$, it follows that $6|1(1+1)(1+2)$.

(ii) Suppose that $6|p(p + 1)(p + 2)$.
Since $(p + 1)(p + 2)(p + 3) = p(p + 1)(p + 2) + 3(p + 1)(p + 2)$, it follows, by Theorem 126d, that

$$6|(p + 1)(p + 2)(p + 3) \text{ if } 6|3(p + 1)(p + 2)$$

But, by the result of Exercise 6, since $p + 1 \in I^+$, it follows that $2|(p + 1)(p + 2)$. So, by Theorem 126f, $2 \cdot 3|(p + 1)(p + 2)3$. So, $6|3(p + 1)(p + 2)$, and, as shown above, it follows that

$$6|(p + 1)(p + 2)(p + 3).$$

Consequently, $\forall_n$ $[6|n(n+1)(n+2) \Rightarrow 6|(n+1)([n+1]+1)([n+1]+2)]$.

*8. For each n, $n(n^2 + 5) = n(n^2 - 1) + 6n = (n - 1)n(n + 1) + 6n$. Obviously, $6 | 1(1^2 + 5)$. For $p > 1$, it follows, by the result of Exercise 7, that $6 | (p - 1)p(p + 1)$. Since, by definition, $6 | 6p$, it follows from Theorem 126d that, for $p > 1$, $6 | (p - 1)p(p + 1) + 6p$. So, for $p > 1$, $6 | p(p^2 + 5)$. Since, by Theorem 104, $p \geq 1$, it follows that, for any positive integer p, $6 | p(p^2 + 5)$. Consequently, $\forall_n\ 6 | n(n^2 + 5)$.

*9. [For the instance '$24 | 1(1^2 - 1)$' of this generalization to be true [or, even, meaningful] the definition of '$|$' must be extended so that 0 belongs to its domain. Students who work starred exercises should have no trouble with this extension.] For p odd, there is a positive integer, say q, such that $p = 2q - 1$. Then,

$$\begin{aligned} p(p^2 - 1) &= (2q - 1)[(2q - 1)^2 - 1] \\ &= (2q - 1)(2q - 2)(2q) \\ &= 4(2q - 1)(q - 1)q \\ &= 4[(q + 1) + (q - 2)](q - 1)q \\ &= 4[(q - 1)q(q + 1) + (q - 2)(q - 1)q]. \end{aligned}$$

Since, by [a slight extension of] the result of Exercise 7, both $(q - 1)q(q + 1)$ and $(q - 2)(q - 1)q$ are divisible by 6, it follows, by Theorem 126d, that their sum is divisible by 6. Hence, by Theorem 126f, 4 times their sum is divisible by 24. So, for p odd, $24 | p(p^2 - 1)$. Consequently, for each odd n, $24 | n(n^2 - 1)$.

*

There are many theorems like those in Exercises 6 - 9 of Part B. One like that of Exercise 7 is '$\forall_n\ 6 | n(n+1)(n+5)$'. The sequel to Exercise 7 is '$\forall_n\ 24 | n(n+1)(n+2)(n+3)$', and the next is '$\forall_n\ 120 | n(n+1)(n+2)(n+3)(n+4)$'. With these, we can prove '$\forall_n\ 30 | n^5 - n$' and 'For each odd n, $240 | n^5 - n$'.

Relations which, like $\le$ and $|$, are reflexive, transitive, and antisymmetric are called *reflexive order relations*. Because, in addition, for each m and n, either $m \le n$ or $n \le m$, $\le$ is called *a total order relation*. Because the analogous generalization does not hold for $|$, the latter is called *a partial order relation*. The relation $\ge$ is also a total order relation. The relation which each person has to himself and to each of his descendants [and to no one else] is a partial order relation. A total order relation can be represented schematically by the order, left to right, of points of a line. Graphical representation of a partial order relation requires branching diagrams like these on page 7-117, or those in genealogical charts.

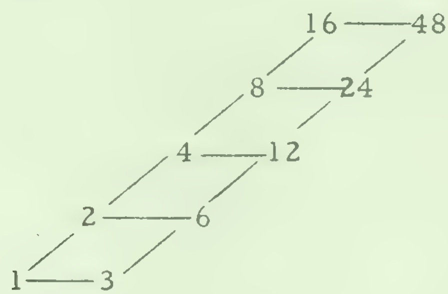
*

Answers for Part A [on page 7-118].

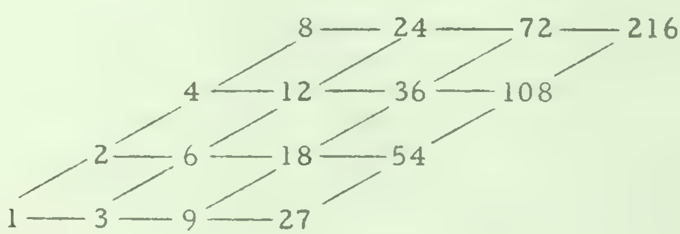
There are infinitely many correct answers. The relation $|$ orders the factors [divisors] of a positive integer linearly if and only if the latter is a power of a prime number.

*

Answer for Part B [on page 7-118].



The divisibility-ordering diagram for the divisors of a positive integer is two-dimensional if and only if the latter is the product of powers of two prime numbers--that is, if and only if it has just two prime factors. This is the case for 18 $[2 \cdot 3^2]$, 54$[2 \cdot 3^3]$, 36$[2^2 \cdot 3^2]$, 108$[2^2 \cdot 3^3]$, and 216$[2^3 \cdot 3^3]$. The diagrams for each of these is part of the diagram for 216:



nswers for Parts A and B are on TC[7-117].

*

nswers for Part C.

$$
\begin{tikzcd}
& & 9 \arrow[rr, dash] & & 45 \arrow[rr, dash] & & 225 \arrow[rr, dash] & & 1125 \\
& 3 \arrow[rr, dash] \arrow[ur, dash] & & 15 \arrow[rr, dash] \arrow[ur, dash] & & 75 \arrow[rr, dash] \arrow[ur, dash] & & 375 \arrow[ur, dash] & \\
1 \arrow[rr, dash] \arrow[ur, dash] & & 5 \arrow[rr, dash] \arrow[ur, dash] & & 25 \arrow[rr, dash] \arrow[ur, dash] & & 125 \arrow[ur, dash] & &
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}
& & 4 \arrow[rr, dash] & & 20 \arrow[rr, dash] & & 100 \arrow[rr, dash] & & 500 \\
& 2 \arrow[rr, dash] \arrow[ur, dash] & & 10 \arrow[rr, dash] \arrow[ur, dash] & & 50 \arrow[rr, dash] \arrow[ur, dash] & & 250 \arrow[ur, dash] & \\
1 \arrow[rr, dash] \arrow[ur, dash] & & 5 \arrow[rr, dash] \arrow[ur, dash] & & 25 \arrow[rr, dash] \arrow[ur, dash] & & 125 \arrow[ur, dash] & &
\end{tikzcd}
$$

[any number which is the product of the square of one prime number and the cube of another]

The diagram for $2^3 \cdot 3^4$ is like that for 216 [given in answers for Part A on TC[7-117]] except that it has an additional slanting row:

$$
\begin{tikzcd}
& & & {} \arrow[r, dash] & 648 \\
& & {} \arrow[r, dash] & 324 \arrow[ur, dash] & \\
& {} \arrow[r, dash] & 162 \arrow[ur, dash] & & \\
{} \arrow[r, dash] & 81 \arrow[ur, dash] & & &
\end{tikzcd}
$$

That for $2^4 \cdot 5^4$ can be obtained by extending that for 500 [given in the answer for Exercise 2 of Part C]--two more horizontal rows and one more slanting row.

(a) 12 (b) 12 (c) 20 (d) 35 (e) 28 (f) 4212

(g) 4212 (h) 8 (i) 4 (j) 9 (k) 18 (ℓ) 7

(m) 72 (n) 18 (o) 18 (p) 90

*

nswer for Part D.

ne divisibility-ordering diagram for the divisors of a positive integer three-dimensional if and only if the latter has just three prime factors.

*

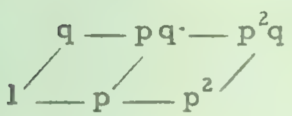
Answers for Part E [which continues on page 7-119].

[The smallest positive integers whose diagrams are of the forms shown are 8 and 6, respectively.]

1. 1 — p — p^2 [4]

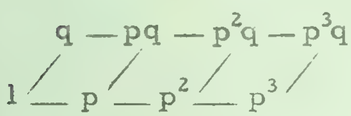
2. 1 — p — p^2 — p^3 — p^4 [16]

3. 1 — p — p^2 — p^3 — p^4 — p^5 [32];

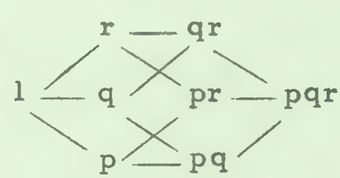


[12]

4. 1 — p — p^2 — p^3 — p^4 — p^5 — p^6 — p^7 [128];

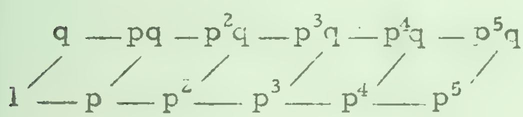


[24];

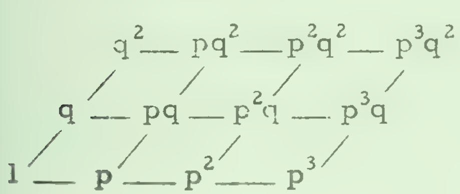


[30]

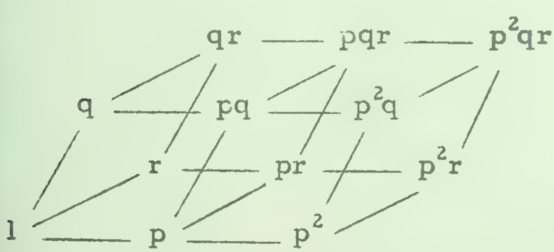
5. 1 — p — p^2 — p^3 — p^4 — p^5 — p^6 — p^7 — p^8 — p^9 — p^{10} — p^{11} [2048];



[96];

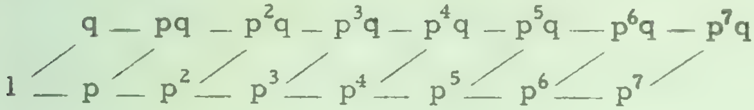


[72];

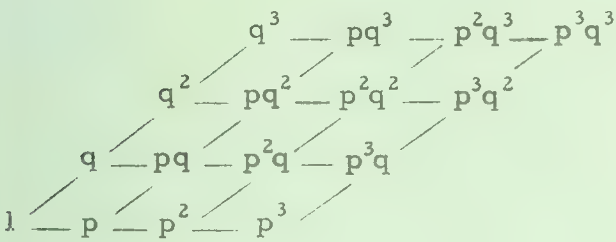


[60]

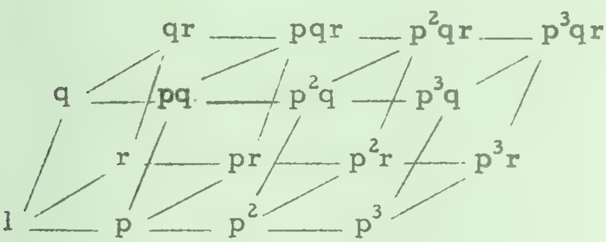
6. $1 — p — p^2 — p^3 \ldots — p^{14} — p^{15}$ [32768];



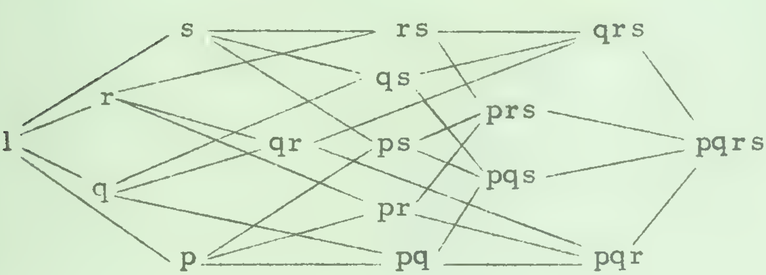
[384];



[216];



[120];



[210]

The remainder of this unit deals with the notion of highest common factor [culminating in Theorem 127 on page 7-122], and with the application of this concept in solving Diophantine problems [pages 7-125 through 7-132].

*

In the analogy between the order relations | and $\leq$, the notion of highest common factor corresponds with that of greatest lower bound. To bring this out, it will be convenient to use 'highest' and 'lowest' in connection with the order relation |, in analogy with 'greatest' and 'least' as these words are used in connection with $\leq$. For example, the set of prime numbers, like any nonempty subset of I^+, has a <u>least</u> member. [The least prime number is 2.] But, the set of prime numbers has no <u>lowest</u> member. [In the ordering by divisibility, all prime numbers are equally low; only the number 1 (which is not a prime) is lower.]

Although a nonempty set of positive integers may fail to have a lowest member, each such set has a highest lower bound. This means that there is a number which divides each member of the set [that is, is a lower bound (with respect to |) of the set] and which is a multiple of each number which has this property [that is, is higher than each other common factor]. This number is called <u>the highest common factor</u> (HCF) of the members of the given set. That each two positive integers do have an HCF is proved on page 7-122. The proof that the members of each nonempty set of positive integers have an HCF is similar.

The notion of <u>highest</u> common factor [or: divisor] should be contrasted with that of <u>greatest</u> common divisor (GCD). It follows easily from Theorems 125 and 126a that the set of common factors of the members of a given set is nonempty and has each member of the given set as an upper bound [with respect to $\leq$]. So, by Theorem 115, the set of common factors of the members of a nonempty subset of I^+ has a greatest member. This means that there is a number which divides each member of the given set and which is <u>greater than</u> each other common divisor--the members of any nonempty subset of I^+ have a greatest common divisor. That this number must be, in fact, a <u>multiple</u> of each common divisor, is not particularly easy to see. On the other hand, once it has been proved, by the method used on page 7-122, that the members of a nonempty subset of I^+ have a <u>highest</u> common factor, it follows at once, by Theorem 126a, that this number is also their <u>greatest</u> common divisor.

HCF(15, 24) = 3; HCF(25, 50) = 25; HCF(21, 27) = 3;

HCF(14, 33) = 1; $\text{HCF}(2^3 \cdot 3^5,\ 2^7 \cdot 3^4) = 2^3 \cdot 3^4$

*

To see that the if-part of the biconditional generalization displayed on page 7-120 "means" that d divides both m and n, consider the if-part of the instance:

$$(d|m \text{ and } d|n) \Longleftrightarrow d|d$$

--that is, the conditional sentence:

$$d|d \Rightarrow (d|m \text{ and } d|n)$$

From this and Theorem 125 follows '$d|m$ and $d|n$'.

*

Suppose that, for each p, $p|m$ and $p|n$ if and only if $p|d_1$, and $p|m$ and $p|n$ if and only if $p|d_2$. It follows [by the substitution rule for biconditional sentences] that, for each p, $p|d_1$ if and only if $p|d_2$. Since, by Theorem 125, $d_1|d_1$, it follows that $d_1|d_2$. Similarly, since $d_2|d_2$, $d_2|d_1$. So, by Theorem 126c, $d_1 = d_2$. Consequently, m and n have at most one HCF.

One says that a given number c is a <u>linear</u> <u>combination</u> of given numbers a and b if and only if there are numbers x and y such that $c = ax + by$. And, one says that c is an <u>integral</u> linear combination of a and b if and only if there are <u>integers</u> x and y such that $c = ax + by$. [Of course, an integral linear combination of integers is, by Theorem 110, an integer. But, the word 'integral' refers to the multipliers rather than to the given numbers.]

*

The explanation asked for eight lines from the bottom of page 7-121 goes as follows: Suppose we can show that the remainder on dividing one integral linear combination of 9672 and 4147 by another is also an integral linear combination of these two numbers. Then, since 9672 and 4147 are both integral linear combinations of 9672 and 4147, and since 1378 is the remainder upon dividing one by the other, it follows that 1378 is an integral linear combination of 9672 and 4147. Furthermore, since 13 is the remainder upon dividing 4147 by 1378, and since both of these are integral linear combinations of 9672 and 4147, so is 13.

*

Before taking up the last paragraph on page 7-121 and the proof on page 6-122, it may be well to remind students that, since

$$\frac{p}{q} = \left[\!\left[\frac{p}{q}\right]\!\right] + \left\{\!\left\{\frac{p}{q}\right\}\!\right\}, \text{ where } 0 \leq \left\{\!\left\{\frac{p}{q}\right\}\!\right\} < 1,$$

it follows that

$$p = \left[\!\left[\frac{p}{q}\right]\!\right]q + \left\{\!\left\{\frac{p}{q}\right\}\!\right\}q, \text{ where } 0 \leq \left\{\!\left\{\frac{p}{q}\right\}\!\right\}q < q.$$

So, $\left[\!\left[\frac{p}{q}\right]\!\right]$ is the <u>partial</u> <u>quotient</u> obtained when one "divides" p by q [using the division-with-remainder algorithm] until one obtains a non-negative remainder which is less than the divisor. The remainder is $\left\{\!\left\{\frac{p}{q}\right\}\!\right\}q$ or, equivalently, $p - \left[\!\left[\frac{p}{q}\right]\!\right]q$.

In line 7 on page 7-122, students are asked to find integers i and j such that

$$13 = 9672i + 4147j$$

The previously proved theorem [last paragraph on page 7-121] guarantees that such integers exist, but the job of finding them can be complicated. The formal technique is discussed on pages 7-123 and 7-124 but it is a good idea to anticipate that work now. We know that both 9672 and 4147 are integral linear combinations of 9672 and 4147:

(1) $$9672 = 9672 \times 1 + 4147 \times 0$$

(2) $$4147 = 9672 \times 0 + 4147 \times 1$$

So, according to our theorem, the remainder, 1378, upon dividing 9672 by 4147, is also an i.ℓ.c. of 9672 and 4147. Since

(*) $$9672 - 4147 \times 2 = 1378,$$

substituting from (1) and (2) into (*) yields:

$$[9672 \times 1 + 4147 \times 0] - [9672 \times 0 + 4147 \times 1] \times 2 = 1378$$

or, more simply:

(3) $$9672 \times 1 + 4147 \times -2 = 1378$$

Equation (3) reveals one pair of integral multipliers. [Of course, we could have found these multipliers directly from (*), but we are preparing for the mechanical procedure shown on page 7-124.] Again, by our theorem, since both 4147 and 1378 are i.ℓ.c.s of 9672 and 4147, so is the remainder, 13, one gets upon dividing 4147 by 1378. By long division,

(**) $$4147 - 1378 \times 3 = 13.$$

Now, we substitute from (2) and (3) into (**) to get:

$$[9672 \times 0 + 4147 \times 1] - [9672 \times 1 + 4147 \times -2] \times 3 = 13$$

or, more simply:

(4) $$9672 \times \underbrace{-3}_{\uparrow\, i} + 4147 \times \underbrace{7}_{\uparrow\, j} = 13$$

*

One way to make sure that students understand the notation in line 13 on page 7-122 is to use an example, say:

$$S = \{p: \exists_i \exists_j \; p = 7i + 9j\},$$

and proceed as follows. Imagine the set of all ordered pairs of integers (i, j)--in other words, think of the number plane lattice. Now, take each of these ordered pairs, multiply 7 by the first component, multiply 9 by the second component, and add the products. The result is an integer [Why?]. The positive results are precisely the members of S. Integral-linear-combining-of-7-and-9 is an operation which maps the number plane lattice into the set of integers. Does this mapping have an inverse? Clearly, it does not because, say,

$$(8, -6) \rightarrow 7 \cdot 8 + 9 \cdot -6, \text{ or } 2$$

and $$(17, -13) \rightarrow 7 \cdot 17 + 9 \cdot -13, \text{ or } 2.$$

[An interesting problem is to find all the ordered pairs which map onto 2. This is an example of linear Diophantine problems which are treated later in the unit.]

Does the set S contain all the positive integers? In this case it does, and if you want to have some fun with your students, challenge them to give you a positive integer for which you can't find an ordered pair which maps onto the given positive integer. If they say 209, you come back with (836, −627). In fact, for each p,

$$(4p, -3p) \rightarrow p.$$

This is so because $(4, -3) \rightarrow 1$. Of course, if you choose as an example:

$$S = \{p: \exists_i \exists_j \; p = 12i + 15j\},$$

you will find that S is a proper subset of I^+. For example, there is no ordered pair of integers which can be mapped under integral-linear-combining-of-12-and-15 onto 2. In fact, S is the set of positive multiples of 3.

3.

$$
\begin{array}{r}
\quad 1 \\
8749\,\overline{)\,11143} \\
\underline{8749}
\end{array}
\quad
\begin{array}{r}
3 \\
2394\,\overline{)\,8749} \\
\underline{7182}
\end{array}
\quad
\begin{array}{r}
1 \\
1567\,\overline{)\,2394} \\
\underline{1567}
\end{array}
\quad
\begin{array}{r}
1 \\
827\,\overline{)\,1567} \\
\underline{827}
\end{array}
$$

$$
\begin{array}{r}
1 \\
740\,\overline{)\,827} \\
\underline{740}
\end{array}
\quad
\begin{array}{r}
8 \\
87\,\overline{)\,740} \\
\underline{696}
\end{array}
\quad
\begin{array}{r}
1 \\
44\,\overline{)\,87} \\
\underline{44}
\end{array}
\quad
\begin{array}{r}
1 \\
43\,\overline{)\,44} \\
\underline{43}
\end{array}
\quad
\text{HCF}(11143,\ 8749) =
\begin{array}{r}
43 \\
1\,\overline{)\,43} \\
\underline{43} \\
0
\end{array}
$$

	1	0
−1	0	1
−3	1	−1
−1	−3	4
−1	4	−5
−1	−7	9
−8	11	−14
−1	−95	121
−1	106	−135
−43	−201	256
	8749	−11143

4.

$$
\begin{array}{r}
\quad 2 \\
4147\,\overline{)\,10672} \\
\underline{8294}
\end{array}
\quad
\begin{array}{r}
1 \\
2378\,\overline{)\,4147} \\
\underline{2378}
\end{array}
\quad
\begin{array}{r}
1 \\
1769\,\overline{)\,2378} \\
\underline{1769}
\end{array}
\quad
\begin{array}{r}
2 \\
609\,\overline{)\,1769} \\
\underline{1218}
\end{array}
$$

$$
\begin{array}{r}
1 \\
551\,\overline{)\,609} \\
\underline{551}
\end{array}
\quad
\begin{array}{r}
9 \\
58\,\overline{)\,551} \\
\underline{522}
\end{array}
\quad
\text{HCF}(10672,\ 4147) =
\begin{array}{r}
2 \\
29\,\overline{)\,58} \\
\underline{58} \\
0
\end{array}
$$

	1	0
−2	0	1
−1	1	−2
−1	−1	3
−2	2	−5
−1	−5	13
−9	7	−18
−2	−68	175
	143	−368

Answers for Part B are on TC[7-125]a.

Answers for Part A.

1.

$$
\begin{array}{r}
3 \\
2387\,\overline{)\,7469} \\
\underline{7161} \\
308
\end{array}
\quad
\begin{array}{r}
7 \\
308\,\overline{)\,2387} \\
\underline{2156} \\
231
\end{array}
\quad
\begin{array}{r}
1 \\
231\,\overline{)\,308} \\
\underline{231} \\
77
\end{array}
\quad
\begin{array}{r}
3 \\
77\,\overline{)\,231} \\
\underline{231} \\
0
\end{array}
$$

$\text{HCF}(7469, 2387) = 77$

	1	0
−3	0	1
−7	1	−3
−1	−7	22
−3	8	−25
	−31	97

[The computation at the right is not asked for, but you may wish to require it. The portion above the dashed line shows that

$$77 = 7469 \cdot 8 + 2387 \cdot -25.$$

The last line shows that

$$0 = 7469 \cdot -31 + 2387 \cdot 97.$$

So, $\dfrac{2387}{7469} = \dfrac{31}{97}$ (see Exercise 1 of Part B).]

2.

$$
\begin{array}{r}
1 \\
4389\,\overline{)\,5320} \\
\underline{4389} \\
931
\end{array}
\quad
\begin{array}{r}
4 \\
931\,\overline{)\,4389} \\
\underline{3724} \\
665
\end{array}
\quad
\begin{array}{r}
1 \\
665\,\overline{)\,931} \\
\underline{665} \\
266
\end{array}
\quad
\begin{array}{r}
2 \\
266\,\overline{)\,665} \\
\underline{532} \\
133
\end{array}
\quad
\begin{array}{r}
2 \\
133\,\overline{)\,266} \\
\underline{266} \\
0
\end{array}
$$

$\text{HCF}(5320, 4389) = 133$

	1	0
−1	0	1
−4	1	−1
−1	−4	5
−2	5	−6
−2	−14	17
	33	−40

To obtain the multipliers of 2618 [the smaller of the two given numbers] one begins with:

$$M_0 = 0, \qquad M_1 = 1$$

[Note that, in order to make it easier to carry out the computations at sight, the table is arranged so that the multipliers M_p are listed in the same line as $-q_{p+1}$.]

Of course, it is completely unnecessary for students to memorize the algorithm in the text, or the shorter algorithm described above. The purpose in discussing the algorithm at all is to show students that one tries to reduce complicated computational procedures to routines in order to dispense with as much thinking as possible in case one has a lot of similar problems to solve. In the case at hand, there really aren't enough problems to justify memorizing the algorithm. [In the case of the long division algorithm learned in grade school, we have an entirely different story.] We could just as well have each student use the techniques he invented to solve the problem in line 7 of page 7-122.

*

Your ablest students may be interested in a still shorter procedure for finding the multipliers for expressing the HCF as an integral linear combination of the two given numbers. This method is given by D. H. Lehmer in "A Note on the Linear Diophantine Equation", *American Mathematical Monthly*, Vol. 48 (1941), pp. 240-246. One lists the partial quotients in *reverse* order, and proceeds, as above, to compute numbers M_p.

	0
3	1
4	3
2	13
2	29
2	71
1	171
	242

Then, $3705 \cdot 171 - 2618 \cdot 242 = (-1)^7 \cdot \mathrm{HCF}(3705, 2618)$.

*

second step in the solution of Example 1, the least nonnegative remainder obtained on dividing 2618 by 1087 is 444, and the corresponding partial quotient is 2. So,

$$2618 - 1087 \cdot 2 = 444.$$

Using the procedure on page 7-121,

$$2618 - 1076 \cdot 2 = (3705 \cdot 0 + 2618 \cdot 1) - (3705 \cdot 1 + 2618 \cdot -1)2$$
$$= 3705\,(0 - 1 \cdot 2) + 2618(1 - -1 \cdot 2).$$

So,

$$444 = 3705 \cdot -2 + 2618 \cdot 3.$$

Continuing in this way, one expresses each remainder as an integral linear combination of 3705 and 2618.

Once understood, the work in the solution of Example 2 can be lessened considerably. One begins by listing in a column the opposites of the partial quotients obtained in the solution of Example 1 [opposites, in order to replace subtractions by additions]. Then, beginning with '1' and '0' in a second column, one computes successively the multipliers 1, −2, 5, −12, 53, −171. Finally, beginning with '0' and '1' in a third column, one computes successively the multipliers −1, 3, −7, 17, −75, 242. The work looks like this:

	1	0
−1	0	1
−2	1	−1
−2	−2	3
−2	5	−7
−4	−12	17
−3	53	−75
	−171	242

The procedure, for either column, is to compute the multipliers for the pth remainder by the recursion equation:

$$M_p = -q_p \cdot M_{p-1} + M_{p-2}$$

To obtain the multipliers of 3705 [the larger of the two given numbers] one begins with:

$$M_0 = 1, \qquad M_1 = 0$$

Students may find the following arrangement of the work in Example 1 more convenient:

$$
\begin{array}{rrrrrrr}
 & 1 & & & & & \\
2618\,) & \overline{3705} & & & & & \\
 & \underline{2618} & 2 & & & & \\
 & 1087\,) & \overline{2618} & & & & \\
 & & \underline{2174} & 2 & & & \\
 & & 444\,) & \overline{1087} & & & \\
 & & & \underline{888} & 2 & & \\
 & & & 199\,) & \overline{444} & & \\
 & & & & \underline{398} & 4 & \\
 & & & & 46\,) & \overline{199} & \\
 & & & & & \underline{184} & 3 \\
 & & & & & 15\,) & \overline{46} \\
 & & & & & & \underline{45} \\
 & & & & & & 1
\end{array}
$$

*

Students may need help in following the solution for Example 2 on page 7-124. The procedure is to use the technique displayed in the third line from the bottom of page 7-121 to express each of the remainders as an integral linear combination of 3705 and 2618. The first two equations in the solution of Example 2 are trivial, but help one to get started. The third line is obtained by noting [either by inspection or from the first step in the solution of Example 1] that 1087 is the least nonnegative remainder obtained upon dividing 3705 by 2618, and that the corresponding partial quotient is 1. So,

$$3705 - 2618 \cdot 1 = 1087.$$

Now, using the procedure on page 7-121, referred to above,

$$\begin{aligned} 3705 - 2618 \cdot 1 &= (3705 \cdot 1 + 2618 \cdot 0) - (3705 \cdot 0 + 2618 \cdot 1)1 \\ &= 3705(1 - 0 \cdot 1) + 2618(0 - 1 \cdot 1). \end{aligned}$$

So,

$$1087 = 3705 \cdot 1 + 2618 \cdot -1.$$

The fourth line in the solution is obtained in a similar manner, using the second and third equations on the right. By inspection, or from the

Answers for Part B of the ☆Exploration Exercises.

1. [(5k, −3k), for any k]
2. [(5k, −3k), for any k]
3. [(17k, −13k), for any k]
4. [(17k, 13k), for any k]
5. [(13k, −17k), for any k]
6. [(1263k, 1322k), for any k]

*

Answers for Part C of the ☆Exploration Exercises.

1. (a) yes (b) no [(5, −3)] (c) yes; yes

*

The discovery to be made in Part C is that stated in Theorem 129 on page 7-129, together with the fact that the antecedent, 'HCF(m, n) = 1', is necessary to the only-if-part of the consequent.

*

2. (a) T (b) F [(4, −5) belongs to the second set but not to the first.]
   (c) T (d) T (e) T (f) F

3. Replace, in the left side, '584' by '292', and '626' by '313'.

4. 67k; −83k

*

For answers for Part D, see COMMENTARY for page 7-126.

Answers for Part B [which is on page 7-124].

[These should be obtained either by dividing out HCFs found in Part A, or by the technique suggested in the COMMENTARY on Exercise 1 of Part A.]

1. $\frac{31}{97}$ 2. $\frac{33}{40}$ 3. $\frac{8749}{11143}$ 4. $\frac{143}{368}$

*

Answers for Part C [top of page 7-125].

1. (−5, 8) 2. (17, −15) 3. (−5, 8)

4. (17, 15) 5. [impossible] 6. (34, −30)

The answers for Exercises 1 and 2 of Part C were obtained by the method of the Solution for Example 2 on page 7-124. The answer for Exercise 3 is derived in an obvious way from that for Exercise 1. Those for Exercises 4 and 6 are obtained in obvious fashions from that for Exercise 2. The equation of Exercise 5 has no integral solutions. For, for all integers i and j, $270i + 170j$ is an even number, but 11 is odd.

Exercises 1 and 2 [and, so, Exercises 3, 4, and 6] have infinitely many correct answers. In fact, for each k, $(-5 + 17k,\ 8 - 27k)$ is a pair of integers which satisfy the equation of Exercise 1, and $(17 + 43k,\ -15 - 38k)$ is a pair of integers which satisfy the equation of Exercise 2.

*

Answers for Part A of the ☆Exploration Exercises.

1. (−3, 4) [(−3 + 10k, 4 − 13k), for any k]

2. (3, −2) [(3 − 10k, −2 + 7k), for any k]

3. (−6, 8) [(−6 + 10k, 8 − 13k), for any k]

4. There is none.

5. (−3, 7) [(−3 + 11k, 7 − 25k), for any k]

6. (2387, −7469) [(31k, −97k), for any k]

*

Part D deals with the question of how many pairs of nonnegative integers satisfy:

$$5x + 10y = 75, \qquad \text{[Exercise 1]}$$

$$5x + 10y = 73, \qquad \text{[Exercise 2]}$$

and:

$$25x + 50y = 375 \qquad \text{[Exercise 3]}$$

The equations for Exercises 1 and 3 are both equivalent to '$x + 2y = 15$'. One solution in integers is (1, 7), and so [since HCF(1, 2) = 1] x and y are integers which satisfy this equation if and only if, for some integer k, $x = 1 + 2k$ and $y = 7 - k$. These integers are nonnegative if and only if $k \geq -\frac{1}{2}$ and $k \leq 7$. So, there are 8 solutions in nonnegative integers-- (1, 7), (3, 6), (5, 5), (7, 4), (9, 3), (11, 2), (13, 1), and (15, 0).

The equation for Exercise 2 has no nonnegative integral solutions because, for any nonnegative integers x and y, $5 \mid 5x + 10y$, but $5 \nmid 73$.

Students can, of course, solve these exercises by inspection, but the analysis just given brings out the import of Part C, and anticipates the development in the pages which follow.

*

Answers for Part D of the ☆Exploration Exercises.

1. eight
2. none
3. eight

[Students may quibble about the answers for Exercises 1 and 3. But, 'nothing but nickels and dimes' admits 15 nickels and no dimes. Had the words 'nothing but' been omitted, one might argue that making change with just 15 nickels was not using nickels and dimes. The word 'only' in the eighth line from the bottom of page 7-127 has the same effect as 'nothing but'.]

Theorem 128 and the two corollaries given above can easily be put into words:

> A number which divides a product, and is prime to one factor, divides the other. A prime divisor of a product divides at least one of the factors. If each of two relatively prime numbers divides a third number then so does their product.

*

We might, on page 7-115, define '|' by:

$$\forall_i \forall_j [i|j \Longleftrightarrow \exists_k\, j = ik]$$

However, had this been done, we would still have been interested principally in the divisibility properties of positive integers. For example, the useful Theorems 126a and 126c become false if 'm' and 'n' are replaced by, say, 'i' and 'j'. [The corresponding theorems are:

$$\forall_i \forall_j [i|j \Rightarrow (i \leq j \text{ or } i \leq -j)]$$

and:

$$\forall_i \forall_j [(i|j \text{ and } j|i) \Longrightarrow (i = j \text{ or } i = -j)]]$$

For this reason, it seems pedagogically more suitable to begin in a small way, and extend the definition as, and when, needed, as is done on page 7-129. However, with the exceptions noted above, Theorems 125 through 129 remain theorems when 'm', 'n', and 'p' are replaced by variables whose domain is I. The proofs of some of the theorems so obtained are somewhat more complicated than those of the original theorems.

Correction.

On page 7-130, line 3 should read:

$$\forall_i \forall_j\ [mi + nj = 0 \iff \exists_k\ (i = nk \text{ and } j = -mk)]$$

Theorem 128 is one of the most important elementary theorems about $|$. As the last paragraph on page 7-129 shows, this theorem is a corollary of Theorem 127. Here is a useful corollary of Theorem 128:

$$\forall_m \forall_n \forall_p\ [(p \text{ is a prime number and } p|mn) \Rightarrow (p|m \text{ or } p|n)]$$

Proof: If p is a prime and does not divide m then p and m are relatively prime. So, by Theorem 128, if $p \nmid m$ and $p|mn$ then $p|n$--that is, if $p|mn$ then ($p|m$ or $p|n$).

*

The inference alluded to by the words '--that is,' in the preceding proof is [essentially] of the form:

$$\frac{(p \text{ and not } q) \Rightarrow r}{p \Rightarrow (q \text{ or } r)}$$

Such inferences can be justified by using the rule of exportation [see page 6-353 of the Appendix to Unit 6] and one of the kinds of inference discussed on TC[7-34]b. Here is how:

$$\cfrac{\overset{*}{p} \qquad \cfrac{(p \text{ and not } q) \Rightarrow r}{p \Rightarrow (\text{not } q \Rightarrow r)}\ \ [\text{Exportation.}]}{\cfrac{\text{not } q \Rightarrow r}{\cfrac{q \text{ or } r}{p \Rightarrow (q \text{ or } r)}\,*}}$$

Inference in the opposite direction can be justified by using the other kind of inference discussed on TC[7-34]b and importation.

*

Another useful corollary of Theorem 128 is the following:

$$\forall_m \forall_n \forall_p\ [(\mathrm{HCF}(m, n) = 1 \text{ and } m|p \text{ and } n|p) \Rightarrow mn|p]$$

Proof: Suppose that $m|p$. Then, for some q, $p = mq$. Suppose that $n|p$. Then, $n|mq$ and, since $\mathrm{HCF}(m, n) = 1$, it follows that $n|q$. So, for some q_o, $q = nq_o$. Consequently, $p = m(nq_o) = (mn)q_o$--that is, $mn|p$.

*

$\frac{i}{37} - \frac{j}{29} = \frac{29i - 37j}{37 \cdot 29}$. So, our problem is to find such integers i and j for which $29i - 37j$ is positive and as small as possible. Since 29 and 37 are relatively prime, there exist integers i and j such that $29i - 37j = 1$. We can't do better than this [since, for all i and j, $29i - 37j \in I$]. The solutions of this last equation are $(-14 + 37k, -11 + 29k)$, for $k \in I$, and the only one which satisfies the required inequations is (23, 18). So, the 23rd red line comes the least distance [1/1073 unit] after one of the green lines [the 18th]. By symmetry [or, by solving '$29i - 37j = -1$'], the 14th red line comes the least distance before one of the green lines.]

4. \$1.00 [If googels cost x¢, goggels y¢, and gaggels z¢, then $4x + 5y - 3z = 25$ and $2x + 4y + 3z = 38$. From this it follows that $2x + 3y = 21$. The integral solutions of this equation are $(42 - 3i, -21 + 2i)$, for $i \in I$. Substituting, one finds that $3z + 2i = 38$. A pair (z, i) of integers satisfies this equation if and only if $z = 2j$ and $i = 19 - 3j$, for some $j \in I$. So, a triple (x, y, z) of integers satisfies the two given equations if and only if $x = -15 + 9j$, $y = 17 - 6j$, and $z = 2j$, for some $j \in I$. Only for $j = 2$ are these integers positive. So, a googel costs 3¢, a goggel costs 5¢, and a gaggel costs 4¢.]

The answer to Exercise 5 of Part D is on TC[7-132].

Answer for Part C.

280 [The ordered pairs $(x, y) \in I \times I$ such that $5x + 7y = p$ are $(3p + 7k, -2p - 5k)$, for $k \in I$. Those whose components are nonnegative integers are those for which $-\frac{3p}{7} \leq k \leq \frac{-2p}{5}$. In order that there be at least 9 such ordered pairs it is <u>necessary</u> that $-\frac{2p}{5} - -\frac{3p}{7} \geq 8$--that is, that $p \geq 8 \cdot 35$. Now, if $p = 8 \cdot 35$ then the solutions whose components are nonnegative integers are those for which $-120 \leq k \leq -112$. Since there are just 9 of these, $8 \cdot 35$ is the sought-for value of 'p'.]

*

Answers for Part D [which continues on page 7-132].

1. 4 [The equation to be solved is '$5i + 10j = 55$' subject to the restrictions '$0 \leq i \leq 7$' and '$0 \leq j \leq 6$'. The solutions of the equation are $(-11 + 2k, 11 - k)$, for $k \in I$. To satisfy the restrictions, $0 \leq -11 + 2k \leq 7$ and $0 \leq 11 - k \leq 6$--that is, $\frac{11}{2} \leq k \leq 9$ and $5 \leq k \leq 11$. So, $6 \leq k \leq 9$, and the problem has four solutions--(1, 5), (3, 4), (5, 3), and (7, 2).]

2. \$14.32 [The equation to be solved is '$100j + i - 350 = 2(100i + j)$', or equivalently, '$199i - 98j = -350$', subject to the restrictions '$0 < i < 100$' and '$0 < j < 100$'. The solutions of the equation are $(-33 \cdot 350 + 98k, -67 \cdot 350 + 199k)$, for $k \in I$. The restrictions imply that $k = 118$.]

3. the 23rd; the 14th [Taking the length of the stick as unit, the distance from the given end to the ith red mark is $\frac{i}{37}$, and the distance to the jth green mark is $\frac{j}{29}$. So, to answer the first question, we want to find positive integers i and j such that $\frac{i}{37} - \frac{j}{29}$ is positive and as small as possible and such that $i \leq 36$ and $j \leq 28$. [If there are several solutions, we want the one for which i is least.] Now,

> Correction. On page 7-131, line 14 should read:
>
> (3′) $\exists_k$ (i = 6k and j = 5k)
>
> and line 16 should read:
>
> (−2 + 6k, −2 + 5k), for k ∈ I.

Answers for Part A.

1. (6400 - 77k, −5400 + 65k), for k ∈ I [By the Euclidean algorithm, 77 · −27 + 65 · 32 = 1. So, (32, −27) is a solution of '65i + 77j = 1'. Hence, (6400, −5400) is a solution of '65i + 77j = 200'. Since HCF(65, 77) = 1, the solutions of '65i + 77j = 0' are the ordered pairs (−77k, 65k), for k ∈ I. Hence, the answer.]

   Having obtained the answer given above for Exercise 1, one can find a simpler answer by noting that (6400 - 77 · 83, −5400 + 65 · 83)--that is, (9, −5)--is a solution of '65i + 77j = 200'. So, the solutions of this equation are the ordered pairs (9 - 77k, −5 + 65k), for k ∈ I.

2. (2175 + 63k, 1725 + 50k), for k ∈ I [This answer, like the first given for Exercise 1, was found by using the Euclidean algorithm to obtain the solution (29, 23) for the equation '50i - 63j = 1'. Like the answer for Exercise 1, it can be put in a simpler form by using it to obtain a "small" solution for '50i - 63j = 75'. Using, for this purpose, −34 as a value for 'k', one sees that (33, 25) is one such solution [(−30, −25) is another]. So, the solutions of the given equation are the ordered pairs (33 + 63k, 25 + 50k), for k ∈ I [and, they are also the ordered pairs (−30 + 63k, −25 + 50k), for k ∈ I].]

3. (−1000 + 19k, 1750 - 33k), for k ∈ I [or: (7 + 19k, 1 - 33k), for k ∈ I]

*

Answers for Part B.

1. none [The equation has no integral solutions.]

2. 4 [The integral solutions are (122 + 5k, −61 - 3k), for k ∈ I. These are pairs of positive integers if $-\frac{122}{5} < k < -\frac{61}{3}$--that is, if $-24 \le k \le -21$.]

*

Answers to Exercises 2, 3, and 4 of Part D are on TC[7-131]b, c.

5. 282 [If w is the number of coins in the sack, and x, y, and z are the numbers of coins carried by the three robbers, respectively, then

$$x + y + z = w,$$

$$\frac{\frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z}{3} + \frac{1}{2}x = \frac{1}{2}w,$$

$$\frac{\frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z}{3} + \frac{2}{3}y = \frac{1}{3}w,$$

and

$$\frac{\frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z}{3} + \frac{5}{6}z = \frac{1}{6}w;$$

x, y, z, and w are positive integers, $2|x$, $3|y$, $6|z$, $6|w$, and $3|\frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z$. From the second and third equations it follows that $3x - 4y = w$; from the second and fourth, that $3x - 5z = 2w$. From these together with the first equation it follows that $x = 33z$ and $47x = 33w$. Hence, $x = \frac{33w}{47}$, $y = \frac{13w}{47}$, and $z = \frac{w}{47}$. Since $z \in I^+$, $47|w$. Since, as we saw earlier, $6|w$, and since 6 and 47 are relatively prime, it follows that $6 \cdot 47|w$. Hence, since $w \in I^+$, $w = 282n$, for some $n \in I^+$. So, $x = 33 \cdot 6n$, $y = 13 \cdot 6n$, and $z = 6n$. For each n, $2|x$, $3|y$, and $6|z$. Now, $\frac{1}{2}x + \frac{1}{3}y + \frac{1}{6}z = 126n$ and, for each n, this is divisible by 3. Hence, all the conditions of the problem are met if [and only if] w, x, y, and z are the indicated multiples of some positive integer n. So, the least number of coins which the sack could have contained is 282. [It might have contained any positive integral multiple of this number of coins.]]

[Exercise 5 is adapted from a problem cited in V. Sanford's A Short History of Mathematics [New York: Houghton Mifflin Company, 1930], page 145.]

the cancellation principle for addition from the apa, the pa0, and the po.]

Suppose that b ☆ a = c ☆ a. By (8), there is a number, say e, such that $\forall_y$ y ☆ e = y. By (10), there is a number, say d, such that a ☆ d = e. Since b ☆ a = c ☆ a, it follows that (b ☆ a) ☆ d = (c ☆ a) ☆ d. So, by (3), b ☆ (a ☆ d) = c ☆ (a ☆ d). Hence, b ☆ e = c ☆ e, and b = c. Consequently, (*).

As pointed out in the COMMENTARY for page 2-60, the basic principles pa0 and po can be replaced by (10) [with '+' in place of '☆']. To see why this is so, let's first show that (1), (3), and (10), together, imply (8).

By (10), for any given number a, there is a number, say e, such that a ☆ e = a. By (10), given a, for any number b, there is a number, say c, such that a ☆ c = b. It follows, by (1), that c ☆ a = b. So, b ☆ e = (c ☆ a) ☆ e. By (3), (c ☆ a) ☆ e = c ☆ (a ☆ e) = c ☆ a = b. Hence, b ☆ e = b. Consequently, (8).

Now, as shown above, (1), (3), (8), and (10) imply (13), and it follows from (13) that there is <u>at most</u> one number x such that $\forall_y$ y ☆ x = y. Combining this result with (8), we see that there is exactly one number x such that $\forall_y$ y ☆ x = y. This being the case, we can introduce a name, say '0', for this number. Doing so, we know that $\forall_y$ y ☆ 0 = y [compare with the pa0]. Also, from (10), we conclude that $\forall_x \exists_z$ x ☆ z = 0. From (13) it follows that, for each x, there is at most one number z such that x ☆ z = 0. So, for each x, there is exactly one number z such that x ☆ z = 0. This being the case, we can introduce an operator, say '−', such that, for each x, −x is <u>the</u> number z such that x ☆ z = 0. So, $\forall_x$ x ☆ −x = 0 [compare with the po].

Consequently, with the cpa [(1)] and the apa [(3)], one can use:

$$\forall_x \forall_y \exists_z\ x + z = y$$

in place of the pa0 and the po [and, for that matter, the ps, also].

and (13) are equivalent. For, in this case, each row in the table for ☆ is a list with the same finite number of entries as the number of members of S. And, to say that each number is listed at least once in a given row [(10)] is equivalent to saying that no number is listed more than once in that row [(13)]. If the domain S is infinite, this argument no longer works and either of (10) and (13) may be true and the other false.

For example, suppose that S is the set of nonnegative integers and that ☆ is addition. In this case, (1), (3), (8), and (13) are true, but (10) is false. The entries in the 2-row of the table for ☆ are: 2, 3, 4, ..., respectively. So, the equation '2 ☆ z = 1', for example, has no solution. The 2-row exhibits a one-to-one mapping of the proper subset {2, 3, 4, ...} on S. To say that there is a proper subset of S which can be mapped in a one-to-one way on S is equivalent to saying that S is an infinite set.

On the other hand, suppose that S is, again, the set of nonnegative integers and that ☆ is the operation of "symmetric differencing"--that is, subtracting the smaller of two numbers from the larger [examples: 3 ☆ 5 = 2 = 5 ☆ 3; 7 ☆ 7 = 0]. In this case, (1), (8), and (10) are true, but (13) is false. This time, the entries in the 2-row are: 2, 1, 0, 1, 2, 3, ..., respectively. So, the equation '2 ☆ z = 1' has two solutions. The 2-row exhibits a many-to-one mapping of S on itself. To say that there is such a mapping is equivalent to saying that S is infinite.

The fact that, in the second of the two examples given above, (3) is false, as well as (13), is to be expected. For, in any case, (1), (3), (8), and (10), together, imply (13). More simply, (3), (8), and (10) imply:

$$(*) \qquad \forall_x \forall_y \forall_z \; [y \star x = z \star x \Longrightarrow y = z],$$

and, obviously, this and (1) imply (13). Here is a derivation of (*) from (3), (8), and (10). [Compare with the derivation, on page 2-65 of Unit 2, of

(4) True, again by checking 27 instances. However, if either 'x', 'y', or 'z' has the value 0 or 1, the checking is trivial. This leaves only 1 instance to be considered.

(5) True; again, 27 instances to check. But those for which 'x' or 'y' has the value 0, and those for which 'z' has the value 0 or 1 are trivial. This leaves 4 instances to be checked individually. [Recalling (1), only 3 instances need checking.]

(6) True, since it is a consequence of (2) and (5), both of which are true.

(7) False; $(1 \oplus 1) \star 1 = 2 \neq 1 = (1 \star 1) \oplus (1 \star 1)$

(8) True, as is seen by inspecting the 0-column of the table for ☆.

(9) True, as is seen by inspecting the 1-column of the table for ⊕.

(10) True because each number ["y"] is listed in each row ["x"] of the table for ☆.

(11) False because 1 is not listed in the 0-row of the table for ⊕.

(12) True because each number is listed in each row other than the 0-row of the table for ⊕.

(13) True because no number is listed twice in any row of the table for ☆.

(14) False because 0 is listed more than once in the 0-row of the table for ⊕. [Of course, '$\forall_{x \neq 0} \forall_y \forall_z \, [x \oplus y = x \oplus z \Rightarrow y = z]$' is true.]

(b) the remainder on dividing xy by 3

*

Some of your students may be interested in exploring the connection between (10) and (13). The generalization (10) says that each equation in 'z' of the form '$a \star z = b$' has at least one solution; (13) says that each such equation has at most one solution. From the comments on (10) and (13) in the answers given above for part (a), it is easy to see that if, as in this exercise, the domain of 'x', 'y', and 'z' is a finite set, S, then (10)

Here is a rough analysis of the kinds of Review Exercises on pages 7-133 through 7-144:

1, 2:	review of basic principles through testing whether they hold for finite number systems
3 - 8:	addition, multiplication, and division-with-remainder algorithms
9:	justifying "computing facts"
10:	simplification, of a kind useful in Unit 8
11, 23:	closure of a set with respect to an operation
12 - 22:	inequations
24 - 26:	recursive definitions
27, 28:	combinatorial problems
29:	discreteness of order of integers
30, 31:	bounds and least members
32:	recursive definitions, induction, inequations
33 - 36:	integral part, fractional part
37 - 41:	divisibility [39 - 41 depend on optional sections.]
42 - 61:	miscellaneous algebra and geometry problems

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Answers for Review Exercises.

1. (a) [All true except (7), (11), and (14).]

   (1) True because the table for ☆ is symmetrical about the diagonal running from upper left to lower right.

   (2) [Similar to (1).]

   (3) True, as can be shown by checking 27 instances. However, if either 'x', 'y', or 'z' has the value 0, the corresponding instances are seen, by inspection of the 0-row or the 0-column of the table for ☆, to be true. This leaves only 8 instances to be checked individually.

2. (a)

☆	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

⚓	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

(b) (1) yes (2) yes (3) no (4) cancellation principle

(5) yes (6) $\forall_x \forall_y \exists_z\ x ☆ z = y$

(7) (i) 3 (ii) 0, 3 (iii) none (8) no

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Students should be intrigued by differences between the number systems of Exercises 1 and 2. The generalization (12) of Exercise 1(a) is true for the system of Exercise 1 but false for that of Exercise 2. Equivalently, there is a left cancellation principle for the operation ⚓ of Exercise 1:

(*) $$\forall_{x \neq 0} \forall_y \forall_z\ [x ⚓ y = x ⚓ z \Longrightarrow y = z],$$

but not for that of Exercise 2. Since, in both cases, ☆ has all the usual properties of addition, (*) is, in any case, equivalent to:

(**) $$\forall_x \forall_w\ [(x \neq 0 \text{ and } x ⚓ w = 0 \Longrightarrow w = 0]$$

[compare with Theorem 56, the 0-product theorem]. That (**) holds for Exercise 1 is due to the theorem:

$$\forall_m \forall_n\ [(3 \nmid m \text{ and } 3 \mid mn) \Longrightarrow 3 \mid n]$$

This is a consequence of Theorem 128 and the fact that 3 is a prime number. That (**) does not hold for Exercise 2 is due to the fact that 6 is not prime.

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Exercises 1 and 2 furnish a basis for introducing many concepts from modern algebra. For example, given a set S and an operation ☆ on S [that is, a mapping of S × S into S], an element e_r of S such that $\forall_x\ x ☆ e_r = x$ is called a right-identity element for ☆ [cf. the pa0--0 is a right-identity element for +]. So, (8) of Exercise 1(a) says that ☆ has a right-identity. An element e_ℓ of S such that $\forall_x\ e_\ell ☆ x = x$ is called a left-identity element for ☆. An operation may have any number of right-identity elements. For example, choose any set S which has at least two members, choose a member $a \in S$, and consider the operation ☆ such that, for each x and y in S, $x ☆ y = a$. Clearly, this operation [which, incidentally, is commutative] has neither a right-identity nor a left-identity. On the other hand, consider any set S and the operation ☆ such that, for each x and y in S, $x ☆ y = x$. This operation [which, incidentally, is associative] has as many right-identity elements as there are members of S, but, if S has more than one member, ☆ has no left-identity. The latter fact is to be expected, for, if an operation ☆ has a left-identity e_ℓ and a right-identity e_r then $e_\ell = e_r$ and ☆ has no other identity element--right or left. [This is easy to prove. Since e_ℓ is a left-identity, $e_\ell ☆ e_r = e_r$ and, since e_r is a right-identity, $e_\ell ☆ e_r = e_\ell$. So, $e_r = e_\ell$. The same argument shows that any right-identity for ☆ is e_ℓ and that any left-identity is e_r.] An element of S which is both a right-identity and a left-identity for ☆ is called the identity element for ☆. So, in particular, (1) and (8) of Exercise 1(a) imply that ☆ has an [unique] identity element [for example, 0 is the identity element for + or, for short, the additive identity].

If an operation ☆ has a right-identity e_r then, given an $a \in S$, an element b of S such that $a ☆ b = e_r$ is called a right-inverse of a for ☆ [with respect to e_r]. For example, the po says that, for each x, $-x$ is a right-inverse of x for + [with respect to 0]--for short, $-x$ is the additive inverse of x. The justification of the definite article and the omission

of 'right-' and 'with respect to 0' comes from a general theorem which says that

> if ☆ is an associative operation on S [(3) of Exercise 1(a)] and has a right-identity e_r such that <u>each</u> $x \in S$ has a right-inverse $\bar{x}$ with respect to e_r
>
> then (1) e_r is a left-identity for ☆ [so, ☆ has a unique identity element],
>
> (2) for each $x \in S$, $\bar{x} ☆ x = e_r$, and
>
> (3) for each $x \in S$, $\bar{x}$ is the only $y \in S$ such that either $x ☆ y = e_r$ or $y ☆ x = e_r$.

[Note that (3) includes the "0-sum theorem": $\forall_x \forall_y [x + y = e_r \Rightarrow y = \bar{x}]$]

To establish this we note, first, that if ☆ is an associative operation and has a right-identity e_r, and c has a right-inverse $\bar{c}$ with respect to e_r, then $\forall_x \forall_y [x ☆ c = y ☆ c \Rightarrow x = y]$. For, suppose that $a ☆ c = b ☆ c$. Then, $(a ☆ c) ☆ \bar{c} = (b ☆ c) ☆ \bar{c}$. Hence, since ☆ is associative, since $c ☆ \bar{c} = e_r$, and since e_r is a right-identity, it follows that $a = b$.

Suppose, again, that ☆ is an associative operation and has a right-identity e_r and that each $x \in S$ has a right-inverse $\bar{x}$ with respect to e_r. To establish (1), above, suppose that $a \in S$. Then, $(e_r ☆ a) ☆ \bar{a} = e_r ☆ (a ☆ \bar{a}) = e_r ☆ e_r = e_r$ and, since $\bar{a} ☆ \bar{a} = e_r$, $(e_r ☆ a) ☆ \bar{a} = a ☆ \bar{a}$. Hence, since $\bar{a}$ has a right-inverse, it follows from the cancellation principle proved above that $e_r ☆ a = a$. Consequently, $\forall_x\, e_r ☆ x = x$--that is, e_r is a left-identity for ☆.

To establish (2), note that $(\bar{a} ☆ a) ☆ \bar{a} = \bar{a} ☆ (a ☆ \bar{a}) = \bar{a} ☆ e_r = \bar{a} = e_r ☆ \bar{a}$. So, as before, $\bar{a} ☆ a = e_r$.

To establish (3), note that if $b ☆ a = e_r$ then, since, as just proved, $\bar{a} ☆ a = e_r$, it follows [by cancellation] that $b = \bar{a}$. On the other hand, if

$a \star b = e_r$ then $\bar{a} \star (a \star b) = a \star e_r$ and, so, $(\bar{a} \star a) \star b = \bar{a} \star e_r$. Hence, if $a \star b = e_r$ then $e_r \star b = \bar{a} \star e_r$ and, so, $b = \bar{a}$.

A set S together with an operation $\star$ on S is, by definition, a group if and only if $\star$ has an identity element with respect to which each member of S has an inverse. The real numbers constitute a group under addition, and the nonzero real numbers constitute a group under multiplication. [This has already been noted on TC[5-264]c of Unit 5.] The result just proved is that an associative operation on S which has a right-identity, with respect to which each member of S has a right-inverse, is a group operation. [Obviously, one might read 'left-' for 'right-' in both places in the preceding characterization of group operations. Oddly enough, if one replaces just one of the two 'right-'s by a 'left-', one no longer has a characterization of group operations.]

Another characterization of group operations is that they are those associative operations $\star$ such that $\forall_x \forall_y \exists_z\ x \star z = y$ [(10) of Exercise 1(a)] and $\forall_x \forall_y \exists_z\ z \star x = y$. That group operations, as defined previously, do have these last two properties is obvious, since $a \star (\bar{a} \star b) = (a \star \bar{a}) \star b = e \star b = b$ and $(b \star \bar{a}) \star a = b \star (\bar{a} \star a) = b \star e = b$. On the other hand, it is not difficult to show that an associative operation on a set S which has these properties has a right-identity with respect to which each member of S has a right-inverse. For, let c be some member of S. Then, by the first of the properties in question, there is an $e \in S$ such that $c \star e = c$. Now, if a is any member of S, there is, by the second property, a $b \in S$ such that $b \star c = a$. Hence, $a \star e = (b \star c) \star e = b \star (c \star e) = b \star c = a$. Consequently, e is a right-identity for $\star$. Since, by the first property, for any $a \in S$, there is an $\bar{a} \in S$ such that $a \star \bar{a} = e$, each member of S has a right-inverse with respect to e.

In conclusion, we record definitions of some additional kinds of algebraic systems:

A ring is a set S together with two operations, ☆ and ⊕, such that ☆ is a commutative group operation on S and ⊕ is both right- and left-distributive with respect to ☆. A ring is commutative if both operations are commutative. [The algebraic systems in Exercises 1 and 2 are commutative rings.]

An integral domain is a commutative ring with unity [that is, for which there is an identity element for ⊕] such that ['0' denoting the identity element for ☆] $\forall_x \forall_y\, [x \oplus y = 0 \Rightarrow (x = 0 \text{ or } y = 0)]$ [or, equivalently, $\forall_x \forall_y \forall_{z \neq 0}\, [x \oplus z = y \oplus z \Rightarrow x = y]$]. [The system of Exercise 1 is an integral domain, but that of Exercise 2 is not. The system consisting of I, +, and × is an integral domain.]

A field is a commutative ring for which ⊕, restricted to the nonzero members of S, is a group operation. [Clearly, each field is an integral domain. The system of Exercise 1 is a field, as are the real number system and the rational number system.]

For additional information on these and related matters you may find the following references helpful:

Neal H. McCoy. Introduction to Modern Algebra.
[Boston: Allyn and Bacon, 1960] 304 p.

Marie J. Weiss. Higher Algebra for the Undergraduate.
[New York: Wiley, 1949] 165 p.

Correction. On page 7-135, line 5b should read:

8. The equation '$6x^4 - x^3 - 82x^2 + 81x + 36 \underbrace{= 0}$' has . . . .

3. (a) $9x^3 + 4x^2 - 3x - 4$
(b) $17x^4 + 3x^3 + 2x^2 - 3x + 10$
(c) $6 + 8y + 8y^2 - 4y^3 - 3y^4$
(d) $2a^3 - 8a^2 + 8a - 12$
(e) $5x^4 + 6x^3 - 11x^2 - 8$
(f) 0

4. (a) $35x^3 - 31x^2 + 11x - 3$
(b) $16x^7 - 26x^5 + 8x^4 + 9x^3 - x^2 - 9x + 3$
(c) $8y^9 + 4y^8 + 42y^6 + 14y^5 + 12y^4 + 55y^3 + 21y$
(d) $35a^7 + 10a^6 + 31a^4 + 34a^3 + 8a^2 + 6a + 8$
(e) $50x^5 + 50x^3 + 20x^2 + 12x + 8$
(f) $20x^7 + 47x^5 + 13x^4 + 4x^3 + 14x^2 - 30x - 8$

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(g) $5x^4 + 2x^2 - 3x + 7$
(h) $2x^3 - 4x + 2$

5. $12x + 15$

6. $3 \cdot 2^3 - 2 \cdot 2^2 + 7 \cdot 2 - 30 = 0$

7. $3x^3 - 4x + 7$

8. $-\frac{1}{3}, \frac{3}{2}$

9. (a) $3 + 2 = 3 + (1 + 1) = (3 + 1) + 1 = 4 + 1 = 5$

(b) $5 - 2 = (4 + 1) - 2 = [(3 + 1) + 1] - 2 = [3 + (1 + 1)] - 2 = (3 + 2) - 2 = 3 = 2 + 1 = (1 + 1) + 1$ Now, by Theorem 82, $1 \in P$. Hence, by (P_3), $1 + 1 \in P$. Hence, by (P_3), $(1 + 1) + 1 \in P$. So, $5 - 2 \in P$.

Correction. On page 7-136, line 9b [part (g) of Exercise 11] should be deleted.

10. (a) $$\frac{q(q+4)(q+5)}{3} + (q+2)(q+5) = \frac{[q(q+4) + 3(q+2)](q+5)}{3}$$

$$= \frac{(q^2 + 7q + 6)(q+5)}{3}$$

$$= \frac{(q+1)(q+5)(q+6)}{3}$$

(b) $$\frac{(q-1)q(q+1)(q+2)}{4} + q(q+1)(q+2) = \frac{[(q-1) + 4]q(q+1)(q+2)}{4}$$

$$= \frac{q(q+1)(q+2)(q+3)}{4}$$

(c) $$\frac{q(q-1)(2q+5)}{6} + (q+1)^2 - 1 = \frac{q(q-1)(2q+5)}{6} + q(q+2)$$

$$= \frac{q[(q-1)(2q+5) + 6(q+2)]}{6}$$

$$= \frac{q(2q^2 + 9q + 7)}{6}$$

$$= \frac{(q+1)q(2q+7)}{6}$$

(d) $$\frac{q(q+1)}{2(2q+1)} + \frac{(q+1)^2}{(2q+1)(2q+3)} = \frac{(q+1)[q(2q+3) + (q+1)2]}{2(2q+1)(2q+3)}$$

$$= \frac{(q+1)(2q^2 + 5q + 2)}{2(2q+1)(2q+3)}$$

$$= \frac{(q+1)(q+2)}{2(2q+3)}$$

(e) $$\frac{q(6q^2+15q+11)}{2} + (3q+4)^2 = \frac{(6q^3+15q^2+11q) + (18q^2+48q+32)}{2}$$

$$= \frac{6q^3 + 33q^2 + 59q + 32}{2}$$

$$= \frac{(q+1)(6q^2+27q+32)}{2}$$

$$= \frac{(q+1)[(6q^2+12q+6) + (15q+26)]}{2}$$

$$= \frac{(q+1)[6(q+1)^2 + 15(q+1) + 11]}{2}$$

11.

	+	×	-	÷
(a)	yes	yes	no	no
(b)	yes	no	no	no
(c)	no	yes	no	yes
(d)	yes	yes	no	yes
(e)	no	no	no	no
(f)	no	yes	no	no

12. (a) (1) Theorems 86b and 86c

(2) yes [Theorem 87 is a consequence of Theorem 86b--it must hold if the latter does.]

(b) (1) Theorem 86b

(2) yes

15. (a) $1 - \frac{1}{p} + \frac{1}{(p+1)^2} < 1 - \frac{1}{p+1}$ if and only if $\frac{1}{(p+1)^2} + \frac{1}{p+1} < \frac{1}{p}$. Since $\frac{1}{(p+1)^2} + \frac{1}{p+1} = \frac{p+2}{(p+1)^2}$, and since $p > 0$, this is the case if and only if $p(p+2) < (p+1)^2$--that is, if and only if $0 < 1$. So, since $0 < 1$, $1 - \frac{1}{p} + \frac{1}{(p+1)^2} < 1 - \frac{1}{p+1}$.

(b) $\frac{1}{p} - \frac{1}{p^2} < \frac{1}{p+1}$ if and only if $\frac{p-1}{p^2} < \frac{1}{p+1}$--that is, if and only if $p^2 - 1 < p^2$. Since this is the case, $\frac{1}{p} - \frac{1}{p^2} < \frac{1}{p+1}$.

(c) $\frac{p}{p^2+1} < \frac{2}{3}$ if and only if $3p < 2p^2 + 2$--that is, if and only if $2p^2 - 3p + 2 > 0$. Since $2p^2 - 3p + 2 = 2(p - \frac{3}{4})^2 + \frac{7}{8}$, and since $2(p - \frac{3}{4})^2 \geq 0$, it follows that $2p^2 - 3p + 2 \geq \frac{7}{8} > 0$.

(d) [Since $\forall_n\ n + 1 \in I^+$, the generalization in part (d) is a consequence of that in part (b).]

16. This amounts to showing [the word 'nonsquare' is unimportant] that $\forall_{x>0} \forall_{y>0}\ x^2 + y^2 > xy$. Now, for $a > 0$ and $b > 0$, $ab > 0$ and, since $2 > 1$, $2ab > ab$. But, $a^2 + b^2 \geq 2ab$. Hence, for $a > 0$ and $b > 0$, $a^2 + b^2 > ab$. Consequently, $\forall_{x>0} \forall_{y>0}\ x^2 + y^2 > xy$. [As a matter of fact, it is nearly as easy to show that $\forall_x \forall_y\ [(x \neq 0 \text{ or } y \neq 0) \Longrightarrow x^2 + y^2 > xy]$. Compare this with Exercise 14(h).]

17. Since $a \leq b$ and $d \geq 0$, it follows that $ad \leq bd$. Hence, $ab + bc \leq bc + bd = b(c + d)$. Since $c + d \leq a$ and $b \geq 0$, it follows that $b(c + d) \leq ab$. So, $ad + bc \leq ab$.

(f) For $a < -2$, $-a > 2$ [Theorems 94 and 17] and, since $2 \geq 0$, $(-a)^2 > 2^2$ [Theorem 98c]. Hence [for $a < -2$], $a^2 - 4 > 0$ [Theorem 84]. Since $2 > 0$, it follows, for $-a > 2$, by Theorem 86c, that $-a > 0$. Hence, for $b > 1$, it follows by the mtpi [and the pnl] that $-ab > -a$. So, since $-a > 0$, it follows, by Theorem 86c, that $-ab > 0$ and, by Theorem 81, that $-ab \neq 0$. Hence, by Theorem 100, since $a^2 - 4 > 0$, $-ab \neq 0$, $-a > 0$, and $-ab > -a$, it follows that $\frac{a^2 - 4}{-a} > \frac{a^2 - 4}{-ab}$. So, by algebra [Theorems 33, 21, and 78], it follows that $\frac{4 - a^2}{a} > \frac{4 - a^2}{ab}$. Consequently, (f).

(h) [In the following, reference to any theorem about > should be interpreted as referring to the corresponding theorem about $\geq$. For example, 'Theorem 97a' means '$\forall_x \; x^2 \geq 0$'.] Suppose that $ab \geq 0$. Since $2 \geq 1$ it follows from the mtpi that $2ab \geq ab$. By Theorem 97b, $a^2 + b^2 \geq 2ab$. So, by Theorem 86c, $a^2 + b^2 \geq ab$. Hence, if $ab \geq 0$ then $a^2 + b^2 \geq ab$. Suppose, now, that $0 \geq ab$. By Theorem 97a, $a^2 \geq 0$ and $b^2 \geq 0$. So, by Theorem 91, $a^2 + b^2 \geq 0$, and, by Theorem 86c, $a^2 + b^2 \geq ab$. Hence, if $0 \geq ab$ then $a^2 + b^2 \geq ab$. Since, by Theorem 86a, $ab \geq 0$ or $0 \geq ab$, it follows that, in any case, $a^2 + b^2 \geq ab$. Consequently, (h).

(i) For $a > 0$ and $b > 0$, $\sqrt{a} > 0$, $\sqrt{b} > 0$, $(\sqrt{a})^2 = a$, and $(\sqrt{b})^2 = b$. Suppose that $a < b$. Then, by Theorem 98b [since $\sqrt{b} \geq 0$ and $(\sqrt{b})^2 > (\sqrt{a})^2$], it follows that $\sqrt{b} > \sqrt{a}$. So, by Theorem 100 [since $b - a > 0$, $\sqrt{b} \neq 0$, $\sqrt{a} > 0$, and $\sqrt{b} > \sqrt{a}$], it follows that $\frac{b - a}{\sqrt{a}} > \frac{b - a}{\sqrt{b}}$. Hence [by algebra and Theorem 94], $\frac{a - b}{\sqrt{b}} > \frac{a - b}{\sqrt{a}}$. So, for $a > 0$ and $b > 0$, if $a < b$ then $\frac{a - b}{\sqrt{b}} > \frac{a - b}{\sqrt{a}}$. Consequently, (i).

13. (a) By Theorem 85, $(c - d)(a - b) < 0$ if and only if $-[(c - d)(a - b)] > 0$ --that is [by Theorem 21] if and only if $-(c - d)(a - b) > 0$. By Theorem 33, this is the case if and only if $(d - c)(a - b) > 0$. Consequently, (a).

(b) For $a > 0$, $a \neq 0$. So, by Theorem 95a, for $a > 0$, $\frac{1}{a}a > ba$ if and only if $\frac{1}{a} > b$. Similarly, for $b > 0$, $\frac{1}{b}b > ab$ if and only if $\frac{1}{b} > a$. Since, for $a \neq 0$, $\frac{1}{a}a = 1$ and, for $b \neq 0$, $\frac{1}{b}b = 1$, and since $ab = ba$, it follows that $\frac{1}{a} > b$ if and only if $\frac{1}{b} > a$. Consequently, (b).

14. (a) F (b) T (c) T (d) F[$\forall_{x \geq 0}\ 2x \geq x$] (e) T (f) T

(g) F[$\forall_{u > -1}\ (u + 2)^2 > u^2$] (h) T (i) T (j) F

Here are proofs for (b), (c), (e), (f), (h), and (i) of Exercise 14:

(b) Suppose that $a - 3 \geq b$. Then [using the atpi], $a - b \geq 3$. Since $3 > 0$, it follows, by [a theorem like] Theorem 92, that $a - b > 0$. Hence, if $a - 3 \geq b$ then $a - b > 0$. Consequently, (b).

(c) Suppose that $a > b$ and $a < c$. Then, by Theorem 86c, $c > b$--in particular, $c \geq b$. Hence, if $a > b$ and $a < c$ then $c \geq b$. Consequently, (c).

(e) For $a \neq 0$, $a^2 > 0$ [Theorem 97a] and, by the mtpi, $a^2 \cdot \frac{1}{a} > a^2 \cdot \frac{1}{a^2}$ if and only if $\frac{1}{a} > \frac{1}{a^2}$. So, $\frac{1}{a^2} < \frac{1}{a}$ if and only if $a > 1$: By algebra, [for $a \neq 0$] $\frac{1}{a^2} = \frac{1}{a}$ if and only if $a = 1$. Hence, for $a \neq 0$, $\frac{1}{a^2} \leq \frac{1}{a}$ if and only if $a \geq 1$. Consequently, (e).

18. By Theorem 97a, $\forall_{x \neq 0}\ x^2 > 0$. Since $-1 < 0$, it follows, by Theorem 86b, that $\forall_{x \neq 0}\ x^2 \neq -1$. Since $0^2 = 0 \neq -1$, $\forall_x\ x^2 \neq -1$--that is, $\forall_x\ x^2 + 1 \neq 0$.

19. By Theorem 97a, and the pm0 [Theorem 15], it follows that $\forall_x\ x^2 \geq 0$. Hence, $(a^2 - b^2)^2 \geq 0$--that is, $a^4 + b^4 \geq 2a^2b^2$. Consequently, $\forall_x \forall_y \ldots$.

20. $\frac{9}{17}$, $\frac{1}{2}$, $\frac{9}{25}$, $\frac{14}{50}$, $\frac{7}{30}$, $\frac{1}{5}$, 0, $\frac{1}{-5}$, $\frac{-6}{29}$, $\frac{2}{-7}$

21. (a) $\{x: x > \frac{11}{8}\}$ (b) $\{x: x < \frac{11}{4}\}$ (c) $\{x: x \leq -11 \text{ or } x \geq 4\}$

(d) $\{x: x > \frac{11}{6} \text{ or } x < \frac{5}{3}\}$ [An easy way to solve part (d) is to notice that the given inequation is equivalent to '$3x - 5 < 0$ or $3x - 5 > \frac{1}{2}$'.]

22. 3 oranges and 5 grapefruit

23. (a) true [Suppose that n is an even number. Then, there is a q such that $n = 2q$. Now, $f(n) = 2q + 12 = 2(q + 6)$. Since $q + 6 \in I^+$, f(n) is an even number.]

(b) true [Suppose that $a \in S$. Then, there is an n such that $a = 2n - 1$. Since $n \geq 1$, $2n - 1 \in I^+$. So, $f(a) = 2n - 1 + 12 = 2n + 12 - 1 = 2(n + 6) - 1$. Since $n + 6 \in I^+$. $f(a) \in S$.]

(c) false [$3 \in S$ but $23 \notin S$]

(d) false [$1 \in S$ but $21 \notin S$]

(e) true [Suppose that $k \in S$. Then, there is a j such that $k = 3j$. So, $f(k) = 3j + 6 = 3(j + 2)$. Since $j + 2 \in I$ and $3 \in I$, $3(j + 2) \in I$. Since $j + 2 \in I$ and $3(j + 2) \in I$, $f(k) \in S$.]

(f) true [Suppose that $f(k) \notin S$. Then, there is a j such that $k - 6 = 3j$--that is, such that $k = 3(j + 2)$. But, $j + 2 \in I$. So, $k \notin S$.]

[Students may notice that parts (e) and (f) have the same answer because each set is the complement of the other with respect to I and each mapping is the inverse of the other. Formally, '$e \in S \Rightarrow f(e) \in S$' is equivalent to '$f(e) \notin S \Rightarrow e \notin S$', the latter is equivalent to '$f(e) \in \widetilde{S} \Rightarrow e \in \widetilde{S}$', and '$\forall_e\ [f(e) \in \widetilde{S} \Rightarrow e \in \widetilde{S}]$' is equivalent to '$\forall_e\ [e \in \widetilde{S} \Rightarrow f^{-1}(e) \in \widetilde{S}]$'.]

(g) true [By Theorems 90 and 86c, $a + 1 > 5$ if $a > 5$.]

(r) false [$2 > 0$ and $5 > 0$ but $2 - 2 \not> 0$ and $5 - 3 > 0$];

false [$-2 < 0$ and $3 > 0$ but $-2 - 2 < 0$ and $3 - 3 \not> 0$];

true [$0 > a > a - 2$ and $0 > b > b - 3$];

false [$2 > 0$ and $-5 < 0$ but $2 - 2 \not> 0$ and $-5 - 3 < 0$]

(s) true [$2(y - 3) - 3(x - 2) = 2y - 3x$. So, if (x, y) belongs to S then so does $(x - 2, y - 3)$.]

(t) true [$2(y - 3) - 3(x - 2) = 2y - 3x$ and $3(x - 2) - 2(y - 3) = 3x - 2y$. So, if $(x, y) \in S$ then so does $(x - 2, y - 3)$.]

(u) false [$4(x - 2) - 5(y - 3) = 4x - 5y + 7$. So, if $4x - 5y = 14$ then $4(x - 2) - 5(y - 3) = 21$, and $21 \neq 7$, and $21 \neq 14$. For example, $(1, -2) \in S$ because $4 \cdot 1 - 5 \cdot -2 = 14$, but $(1 - 2, -2 - 3) \notin S$ because $4 \cdot -1 - 5 \cdot -5 = 21$.]

(v) true [$S = \emptyset$. So, any mapping maps S into itself.]

(w) true [$4(x - 2) - 5(y - 3) = 4x - 5y + 7$. So, if, for some p, $4x - 5y = 7p$ then $4(x - 2) - 5(y - 3) = 7(p + 1)$. Since $p + 1 \in I^+$, it follows that if $(x, y) \in S$ then $(x - 2, y - 3) \in S$.]

(x) true [If $y > |x|$ then $2y > 2|x| \geq |x| = |-x|$. So, if $(x, y) \in S$ then $(-x, 2y) \in S$.]

(y) true [$\left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2 = x^2 + y^2$; $(x, y) \in S \Rightarrow \left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right) \in S$]

(z) true [$\left(\frac{x+y}{\sqrt{2}}\right)^2 - \left(\frac{x-y}{\sqrt{2}}\right)^2 = 2xy$ and $\frac{x+y}{\sqrt{2}} \cdot \frac{x-y}{\sqrt{2}} = \frac{x^2-y^2}{2}$. So, if $xy = 0$, $\left(\frac{x-y}{\sqrt{2}}\right)^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2$, and if $y^2 = x^2$, $\frac{x+y}{\sqrt{2}} \cdot \frac{x-y}{\sqrt{2}} = 0$. Hence, if $(x, y) \in S$ then so does $\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$.]

[In parts (y) and (z) the mapping is a counter-clockwise rotation through 45° followed by a reflection in $\{(x, y): y = x\}$.] [In parts (r) through (z), at least, students should sketch graphs.]

(h) false $[2 \in S$ but $4 \notin S]$

(i) true $[1 < a < 3 \Longrightarrow 2 < a + 1 < 4 \Longrightarrow 1 < \frac{a+1}{2} < 2 < 3]$

(j) true $[\frac{1+1}{2} = 1]$

(k) false $[-1 \in S$ but $\frac{1 + -1}{2} \notin S]$ [For what value of 'k' is it the case that $\forall_e\ e \in S \Longrightarrow f(e) \in S$, where $S = \{x\colon x < k\}$?]

(ℓ) true [Suppose that $a \in S$. Then, there are integers p and $q \neq 0$ such that $qa = p$. Then, $a = p/q$. So, $f(a) = \frac{1 + p/q}{2} = \frac{q+p}{2q}$. Since $q \neq 0$, $2q \neq 0$. But, $2q \in I^+$ and $q + p \in I^+$. So, $f(a) \in S$. [Actually, the restriction '$q \neq 0$' is not necessary in view of the domain of 'q'.]]

(m) true [Suppose that $f(a) \notin S$. Then, there is a $q \neq 0$ such that $q \cdot f(a) \in I$--that is, $q(1 + a)/2 \in I$. So, since $[q(1 + a)/2]2 \in I$, $q(1 + a) \in I$; since $q(1 + a) - q \in I$, $qa \in I$. Therefore, $a \notin S$.]

[Parts (ℓ) and (m) have to do with rational numbers and irrational numbers, respectively. For (ℓ), note that the average of two rational numbers is rational; for (m), that the average of a rational number and an irrational number is irrational.]

(n) true $[b + 2 > b > 0]$;
true $[b + 2 > b > 0]$;
false $[-3 < 0$ and $-2 < 0$ but $-3 < 0$ and $-2 + 2 \nless 0]$;
false $[3 > 0$ and $-2 < 0$ but $3 > 0$ and $-2 + 2 \nless 0]$

(o) true $[(7, b) \in S \Longrightarrow (7, b + 2) \in S]$

(p) true $[a < b < b + 2$ and $b + 2 = 2i + 2 = 2(i + 1)]$

(q) true $[4y - (2x)^2 = 4(y - x^2)$. Since $4 > 0$, it follows, by Theorem 84, that if $(x, y) \in S$ then $(2x, 4y) \in S$.]

24. (a) $\begin{cases} C_1 = 1 \\ \forall_n\, C_{n+1} = C_n + 4 \end{cases}$

(b) $\forall_n\, C_n = 4n - 3$;

(i) $4 \cdot 1 - 3 = 1 = C_1$

(ii) Suppose that $C_p = 4p - 3$. Since, by the recursive definition, $C_{p+1} = C_p + 4$, it follows that $C_{p+1} = (4p - 3) + 4 = 4(p+1) - 3$. Hence, $\forall_n\, [C_n = 4n - 3 \Longrightarrow C_{n+1} = 4(n + 1) - 3]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n\, C_n = 4n - 3$.

25. (a) $(6n + 1)$

(b) $\forall_n\, D_n = n(3n - 2)$ [The derivation is like that in Exercise 24(b).]

Exercises 26, 27, and 28 are on page 7-141.

26. (a) 8; 17; 6

(b) $\forall_n\, f_n = 3n - 1$;

(i) $f_1 = 2 = 3 \cdot 1 - 1$

(ii) Suppose that $f_p = 3p - 1$. Since, by the r.d., $f_{p+1} = f_p + 3$, it follows that $f_{p+1} = (3p - 1) + 3 = 3(p + 1) - 1$. Etc.

27. (a) $\frac{n(n - 1)}{2}$ (b) 35

(c) $C(n, 4)$ [For each 4 of the n points there is just one pair of chords which intersect inside the circle and which have these points as end points.

28. (a) 36 $[C(7 + 2, 2)]$ (b) 15 $[C(4 + 2, 2)]$

[Answers for Exercises 26, 27, and 28 are on TC[7-140].]

29. $\frac{135}{24} \notin I$. So, $n \geq \frac{135}{24}$ if and only if $n > \frac{135}{24}$. By Theorem 118b, this is the case if and only if $n > \left[\!\left[\frac{135}{24}\right]\!\right] = 5$.

30. [There are many correct answers. The set P, for example, satisfies both (a) and (b).]

31. yes, $-\frac{1}{4}$; yes, 0; yes, 0 [Since $\frac{1-p}{p^2} = \frac{1}{p^2} - \frac{1}{p} = \left(\frac{1}{p} - \frac{1}{2}\right)^2 - \frac{1}{4}$, it follows that $\forall_n \frac{1-n}{n^2} \geq -\frac{1}{4}$. So, $-\frac{1}{4}$ is a lower bound of the given set. Since $\frac{1-2}{2^2} = -\frac{1}{4}$, it follows that $-\frac{1}{4}$ belongs to the given set. Hence, $-\frac{1}{4}$ is the least member of the given set. Since $\forall_n [n \geq 1$ and $n^2 > 0]$, it follows that each member of the set is nonpositive--that is, 0 is an upper bound of the set. Since 0 belongs to the set, it follows that 0 is the greatest member of the set. Hence 0 is the least upper bound of the set.]

32. (a) (i) $g_1 = 2 \geq 2$

(ii) Suppose that $g_p \geq 2$. By the recursive definition, $g_{p+1} = 3g_p$. So, $g_{p+1} \geq 3 \cdot 2$. Since [by Theorem 104] $3 \geq 1$, it follows that $3 \cdot 2 \geq 2$. So, $g_{p+1} \geq 2$. Hence, $\forall_n [g_n \geq 2 \Rightarrow g_{n+1} \geq 2]$.

(iii) From (i) and (ii) it follows, by the PMI, that $\forall_n g_n \geq 2$.

(b) (i) $g_4 - f_4 = 54 - 53 \geq 1$

(ii) Suppose that $g_p - f_p \geq 1$. By the recursive definitions, $g_{p+1} - f_{p+1} = 3g_p - 2f_p - 3 = g_p + 2(g_p - f_p) - 3 \geq 2 + 2 \cdot 1 - 3 = 1$. Hence, $\forall_n [g_n - f_n \geq 1 \Rightarrow g_{n+1} - f_{n+1} \geq 1]$.

(iii) From (i) and (ii) it follows, by Theorem 114, that . . . .

33. (a) 5 (b) -10 (c) 9.28 (d) 0

34. $(0, 0)$, $(\frac{4}{3}, 1)$, $(\frac{8}{3}, 2)$

35. (a) $\{x: x < 4\}$ (b) $\{x: \{\!\{x\}\!\} \geq 0.11\}$

36. (a) For any real number a, $[\![a]\!] \leq a < a + \frac{1}{2}$ and, by Theorem 118a, since $[\![a]\!] \in I$, it follows that $[\![a]\!] \leq [\![a + \frac{1}{2}]\!]$. Consequently, ...

(b) By Theorem 118c, since $[\![a]\!] + 1 \in I$, it follows that $[\![a]\!] + 1 \geq [\![a + \frac{1}{2}]\!]$ if and only if $[\![a]\!] + 2 > a + \frac{1}{2}$. This is the case because $[\![a]\!] + 1 > a$. Consequently, $\forall_x$ . . . .

37. Since $p \mid p^2$, it follows from Theorem 126e that if $p \mid p^2 + 1$ then $p \mid 1$. So, by Theorem 126a, $p \leq 1$. Since, by Theorem 104, $p \geq 1$, it follows, by Theorem 93, that $p = 1$. Hence, if $p \mid p^2 + 1$ then $p = 1$. On the other hand, by Theorem 125, $1 \mid 1^2 + 1$. So, $\forall_n [n \mid n^2 + 1 \Longleftrightarrow n = 1]$.

38. (a) $20 [4 \cdot 5]$ (b) $16 [4 \cdot 4]$ (c) $27 [3 \cdot 3 \cdot 3]$

39. $(1 + 25k, 1 + 31k)$, for $k \in I$

40. 2

41. 18 cows and 203 chickens

42. $\frac{1}{2}$¢

43. (a) $\dfrac{4(x + y - 1)(x - y)}{x^2 - 4y^2}$. (b) $-\dfrac{3x(2y + 1)}{y}$

Correction. On page 7-148, line 2b should end:

$$--- = \frac{xv - uy}{yvz} \qquad [2\text{-}100]$$

44. $\frac{3}{4}$, $\frac{5}{6}$ [Point out that one can save work by using the quadratic formula to obtain '$\frac{1}{2x-1} = 2$ or $\frac{1}{2x-1} = \frac{3}{2}$' and, from this, '$2x - 1 = \frac{1}{2}$ or $2x - 1 = \frac{2}{3}$'. Etc.]

45. $\frac{s+1}{s}$ [$\frac{s+1}{s} - \frac{r}{r+1} > 0$, since $r > 0$ and $s > 0$. So, by Theorem 84, $\forall_{r>0} \forall_{s>0}\ \frac{s+1}{s} > \frac{r}{r+1}$.]

46. $22 + 2\sqrt{61}$; $\sqrt{61}$

47. 3

48. 6 [A box of plums occupies 1/10 of the total load and a box of cherries occupies 1/15. So, $(x/10) + (x/15) = 1$.]

49. (a) 9600 (b) 10

50. (a) 2 (b) $\frac{12}{65}$ (c) $\frac{14}{23}$

51. 4×3

52. (a) $\frac{1}{2}$ (b) 2

53. 3/2 square feet

54. (a) $\frac{41}{60}$ (b) $\frac{3m}{5}$ (c) $\frac{1}{300}$

55. $n - 1$ [$n - 1$ people are to be eliminated--one in each match.]

56. about 2940 miles

57. 14 miles

58. [any four consecutive integers]

59. $\left(\frac{1}{3}, -\frac{1}{2}\right)$

60. 60 square inches

61. 10000